Multi-parameter mechanism design under budget and matroid constraints ☆

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Abstract

The design of truthful auctions that approximate the optimal expected revenue is a central problem in algorithmic mechanism design. 30 years after Myerson's characterization of Bayesian optimal auctions in single-parameter domains, characterizing multi-parameter domains and providing efficient mechanisms for them still remains a very important unsolved problem. Our work improves upon recent results in this area, introducing new techniques for tackling the problem, while also combining and extending recently introduced tools.

In particular we give the first approximation algorithms for Bayesian auctions with multiple heterogeneous items when bidders have additive valuations, budget constraints and general matroid feasibility constraints.

Keywords: Bayesian Mechanism Design, Auctions, Approximation algorithms, Matroids

1. Introduction

Assume n bidders are competing for m items. Each bidder i has a private valuation $v_{ij} \geq 0$ for item j, drawn from a publicly known distribution. Assume further there is either an *individual* matroid $\mathcal{M}_i$ for each bidder i such that each bidder can only receive an independent set of items (the *individual matroid case*) or a global matroid $\mathcal{M}$ for *all* bidders (the *global matroid case*) such that the set of items assigned to the bidders should be an independent set. What is the optimal revenue maximizing auction?

In his seminal paper [17] Myerson gave a complete characterization of the optimal auction for the case $m = 1$ if the distributions of valuations are uncorrelated. Papadimititriou and Pierrakos [18] recently showed that for $n > 2$ bidders with correlated distributions finding the optimal (dominant strategy incentive compatible) deterministic auction is NP-hard, even if $m = 1$. Thus, one of the main open questions in this area is to deal with multiple items, i.e., the case of $m > 1$, when the bidders' distributions are uncorrelated. This is the problem we study in this paper together with matroid and budget constraints.

Truthfulness. For any mechanism there are various criteria for evaluation. One criterion is which notion of *truthfulness* or *incentive compatibility* it fulfills. Every definition of truthfulness involves some notion of *profit* or *optimality* for a bidder. In our setting we

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assume bidder i receives a set S_i and has to pay p_i for it. Then the *profit* of bidder is $\sum_{j\in S_i} v_{ij} - p_i$. An *optimal* outcome for a given bidder is an outcome that maximizes his profit. We distinguish three notions of truthfulness. (1) A mechanism is *universally truthful* if truth-telling is optimal for the bidder even if he knows the valuations of the other bidders and the random choices made by the mechanism[1]. (2) A mechanism is *truthful in expectation* if revealing the true value maximizes the *expected* profit of every bidder, where the expectation is taken over the internal random coin flips of the mechanism. (3) If a prior distribution of the bidders' valuations is given, then a mechanism is *Bayesian incentive compatible (BIC)* if revealing the true value maximizes the *expected* profit of every bidder, where the expectation is over the internal random coin flips of the mechanism *and* the valuations of the *other* bidders.

Matroid constraints A matroid is a structure that generalizes linear independence in vector spaces, some of our results also hold for an even more general setting downward closed environments (every subset of an independent set is also independent). Unfortunately we know very few algorithms for non-downward closed environments. An auctions that allocates all items is an interesting example of a non-downward closed constraint. On the other hand apart from their mathematical elegance, matroid constraints have interesting economical interpretations: a key example of a matroid domain is the unit-demand domain corresponding to transversal matroids, in which there is a finite set of goods for sale and each agent wants to receive one good from a specified subset of the set of all goods, while capacity constraints correspond to uniform matroids.

Definition 1 (Matroid)**.** *A matroid M is defined on a finite ground set $E(M)$ and a collection of subsets of E are said to be independent. The family of independent sets is denoted by $\mathcal{I}(\mathcal{M})$ and we usually refer to a matroid M listing its ground set and its family of independet sets: $M = (E, \mathcal{I})$. For M to be a matroid, $\mathcal{I}$ must satisfy two main axioms:*
(I_1) If $X \subseteq Y$ and $Y \in \mathcal{I}$ then $X \in \mathcal{I}$,
(I_2) If $X \in \mathcal{I}$ and $Y \in \mathcal{I}$ and $|Y| > |X|$ then $\exists e \in Y \setminus X : X \cup \{e\} \in \mathcal{I}$.

Axiom I_2 says that if X is independent and there exists a larger independent set Y then X can be extended to a larger independent set by adding an element of $Y \setminus X$. It also implies that all maximal indenpendent sets have the same cardinality.

Definition 2. *A* Uniform matroid *is a matroid in which a subset of the elements is independent if and only if it contains at most k elements.*

A Graphical Matroid *is a matroid whose independent sets are the forests in a given undirected graph. In our setting the items correspond to edges of the graph, and the set of items assigned should not contain any cycles.*

Optimal or approximate. The revenue of a mechanism is the sum of the payments collected by the auctioneer $\sum_i p_i$. If a mechanism returns the maximum revenue out of all mechanisms fulfilling a certain type of constraints (e.g. all BIC mechanisms), it is an *optimal* mechanism. If it returns a fraction k of the revenue of the optimal mechanism, we call it a *k-approximation.*

Value distributions. If a prior distribution on the bidders' valuations is assumed then there are also multiple cases to distinguish. In the *correlated bidders setting* the value distributions of different bidders can be correlated. Except for [11] and [18], all prior work

[1] This is independent of whether a distributions of the valuations are given or not.

and also our work assumes that the distributions of different bidders are uncorrelated. We further distinguish the case that for each bidder the distributions of different items are uncorrelated (*the independent items case*), and the case that the value distributions of the same bidder for different items to be correlated (*the correlated items case*). There is strong evidence that it is not possible to design an optimal DSIC mechanism for the correlated items case [3]: Even if there is just *one unit-demand* bidder, but his valuations for the items are correlated, the problem of assigning the optimal item to the bidder can be reduced to the problem of unlimited supply *envy-free pricing* with m bidders [15]. For the latter problem the best known mechanism is a logarithmic approximation and there is strong evidence that no better approximation is possible [5].

Running time model. A final criterion to evaluate a mechanism is whether it runs in time polynomial in the input size. Of course this depends on how the input size is measured. We use the model used in [3], where the running time has to be polynomial in n, m, and the support size of the valuation distributions. All the results we list take polynomial time in this model.

Ellipsoid algorithm The problem being considered by the ellipsoid algorithm is: Given a bounded convex set $P \in \mathbf{R}^n$ nd $x \in P$. Linear programming on the other had can be reduced to the problem of finding an x in $P = \{x \in \mathbb{R}^n : Cx \leq d\}$.

Separation Oracle. To run the ellipsoid algorithm, we need to be able to decide, given $x \in \mathbf{R}^n$ either decide whether $x \in P$ or find a violated inequality. What we need is a separation oracle for P: Given x

Related work. *Correlated bidders.* Dobzinski, Fu, and Kleinberg [11] gave an optimal *truthful-in-expectation* mechanism for $m \geq 1$ in the *correlated* bidders and items setting but *without* any matroid or budget constraints.

Uncorrelated bidders. Chawla et al. [8] studied the case $m \geq 1$ with a universal matroid constraint and general valuation distributions, but with only *unit-demand* bidders *without* budget constraints. For a variety of special matroids, like uniform matroids and graphical matroids, they gave constant factor approximations. Very recently, Chawla et al. [9] gave a constant factor approximation for general matroid constraints *with* budgets, but again only with *unit-demand* bidders. Bhattacharya et al. [3] studied the case of individual uniform matroid constraints and budget constrained bidders. For the correlated items case they presented a BIC mechanism whose revenue is within a factor 4 of the optimal BIC mechanism. Their mechanism is truthful only in expectation. For the independent items case if the valuations additionally fulfill the *monotone hazard rate assumption (MHR)* (see Section 4) they gave a DSIC mechanism that achieves a constant factor of the revenue of the optimal BIC mechanism. For the independent items case where the valuations do *not* fulfill MHR they gave a DSIC mechanism that achieves an $O(\log L)$ approximation of the revenue of the optimal DSIC mechanism and they showed that no better posted-price (defined below) mechanism exists. Here L is the maximum value that any bidder can have for any item.

Independently of our work Alaei [1] also gave a (different) 2-approximate BIC mechanism for the case of correlated items, but only for the case of uniform matroid constraints. He also developed a very interesting general technique for expanding single buyer mechanisms to many buyers, improving the approximation factors from [3], but only for the more restricted case when there are no matroid or capacity constraints.

There have also been a lot of interesting recent developments in baysian mechanism design after the appearence of the conference version of our paper. Daskalakis and Weinberg [10] give almost optimal mechanisms for the case when either the items or the bidders are i.i.d. and there exist budget and uniform matroid constraints.

Table 1: **Independent distributions case:** A summary of our results and previous results from the work of Bhattacharya et al. [3]. All approximation ratios are with respect to LP3 and LP3 achieves an $8e^2$ approximation of the optimal mechanism.

	Individual matroids	Global matroid
General matroids	$O(\log L)$	$O(\log L)$
Uniform matroids	(previous: $O(\log L)$ [3])	$O(\log L)$
General matroids & MHR	$O(\log m), O(k)$	$O(\log m), O(k)$
Uniform matroids & MHR	9, (previous: 24 [3])	9
Graphical matroids & MHR	32/3	64 with budgets and 3 without

Kleinberg and Weinberg [16] give an alternative to prophet inequalities [20] that can be also generalized for matroids or matroid intersections, providing constant approximation factors for unit-demand multi-parameter settings generalizing the results of [8]. Cai et al. [7] [6] and Alaei et al. [2] compute the probability that the agent will be served as a function of the type he reports for the single-item case and use this solution to give tractable revenue-optimal multi-item auctions.

Matroid constraints for auctions are also studied in [4] [13], but in a different context, that of ascending auctions.

Our results. We use the same model as Bhattacharya et al. [3], i.e., both matroid and (public) budget constraints. We improve upon their work since they studied only individual matroid constraints where the matroid is a uniform matroid. Specifically we show the following results. (1) For the correlated items with individual matroid constraints case we present a BIC mechanism whose revenue is within a factor 2 of the optimal BIC mechanism. In [3] a 4-approximation for uniform matroids was given. (2) For the independent items case we study general matroid constraints, both in the global and the individual setting. Our mechanisms are DSIC sequential posted price mechanisms [2]. Our results are summarized in table 1: Our results on global matroid constraints are a generalization of the work by Chawla et al. [8, 9]. They gave a constant approximation for global uniform matroids and global graphical matroids, and recently in [9] also for general matroids (in [9] however the authors do not provide a polynomial-time algorithm), but only for the special case when the bidders are *unit demand* (which is reducible to the single-parameter problem). We give constant approximations for *bidders with arbitrary demands* (i.e. for the general multi-parameter problem) for the case of global uniform and graphical matroids however with the assumption that the valuation distributions fulfill the monotone hazard rate condition. All our results take polynomial time.

Our tools and techniques. The basic idea of [3] and [11] is to solve a linear program to determine prices and assignment probabilities for bidder-item pairs. We use the same general approach but extend the linear programs of [3] by suitable constraints that are (i) "strong enough" to enable us to achieve approximation ratios for *general* matroids, but also (ii) "weak enough" so that they can still be solved in polynomial time using the *Ellipsoid*

[2] In [3] a *sequential posted-price (spp)* mechanism is defined as follows: The bidders are considered sequentially in arbitrary order and each bidder is offered a subset of the remaining items at a price for this item and bidder; the bidder simply chooses the profit maximizing bundle out of the offered items. These mechanisms have the advantage of being more practical as they do not require from the bidders to report their valuations but only to take or leave items with posted prices. Experimental evidence also suggests that in spp players tend to act more rationally and they are more likely to participate.

method with a polynomial-time separation oracle. In the correlated items case the results of this new LP together with a modified mechanism and a careful analysis lead to the *improved approximation factor over [3], even with general matroid constraints*.

In the independent items case Bhattacharya et al. [3] used Markov inequalities to reason that uniform matroid constraints and budget constraints reduce the expected revenue only by a constant factor. This approach, however, exploits certain properties of uniform matroids and cannot be generalized to graphical or general matroids. Thus, we extended ideas from Chawla et al [8] to develop different *techniques to deal with non-uniform matroid constraints*: (1) For graphical matroids we combine a graph partitioning technique and prophet inequalities [20]. (2) For general matroids we use Lemma 3 (see also Theorem 10 in [8] for a rather simmilar in nature argument) together with a bucketing technique. The lemma says roughly that if a player is asked the same price for all items then the matroid constraints reduce the expected revenue by at most a factor of 2 in the approximation. As we show it holds both in the global as well as in the individual matroids setting. In combination with a bucketing technique that partitions the items into buckets so that all items in the same bucket have roughly the same price the lemma allows us to tackle general matroid constraints in all of our non-constant approximation algorithms. The generality of the lemma makes it very likely that it is further applicable.

We also develop a new way to deal with *budget constraints* that simplifies the proofs and enables us to *improve the approximation factors*, e.g. for uniform matroids from 24 [3] to 9.

The paper is organized as follows. The next section contains all necessary definitions. Section 3 presents the result for correlated, Section 4 for independent valuations.

2. Problem Definition

There are n bidders and a set J of m distinct, indivisible items. Each bidder i has a private valuation $v_{ij} \geq 0$ for each item j drawn from a publicly known distribution $\mathcal{D}_{i,j}$. Additionally each bidder has a budget B_i and cannot be charged more than B_i. We assume that $B_i \geq 8$. (We conjecture that this assumption might not be necessary but our current proof needs it.) If bidder i receives a subset S_i of items and is charged p_i for it then the *profit* of bidder i is $\sum_{j \in S_i} v_{ij} - p_i$. Bidders are *individually rational*, i.e. bidder i only selects S_i if his profit in doing so is non-negative. A bidder is *individually rational in expectation* if his expected profit is non-negative. The goal of the mechanism is to maximize its *revenue* $\sum_i p_i$ under the constraint that $p_i \leq B_i$ for all i, that all bidders are individually rational or individually rational in expectation, and that each item can be sold only once. Additionally there are matroid constraints on the items. We analyze two types of matroid constraints: In the *universal matroid constraint* problem there exists *one* matroid $\mathcal{M}$ such that $\cup_i S_i$ has to be an independent set in $\mathcal{M}$. In the *individual matroid constraint* problem there exists one matroid $\mathcal{M}_i$ *for each* bidder i such that S_i has to be an independent set in $\mathcal{M}_i$.

Assumptions. We make the same assumptions as in [3, 11] (1) For all i and j the number of valuations with non-zero probability, i.e., the support of $\mathcal{D}_{ij}$ is finite and non-zero only at rational numbers. The running time of our algorithms is polynomial in n, m, and the size of the support of $\mathcal{D}_{ij}$, for all bidders i and items j, i.e., in the *size of the input.* (2) The random variable v_{ij} takes only rational values and that there exists an integer L polynomial in the size of the input such that for all i, j, $1/L \leq v_{ij} \leq L$. (3) For each of the matroids $\mathcal{M}_i$ if given a subset S of J we can in time polynomial in the size of the input compute $rank_{\mathcal{M}_i}(S)$ and determine whether S is independent in $\mathcal{M}_i$ or not.

Here we study the setting in which the distribution of the valuations of a *fixed* bidder for different items can be arbitrarily correlated, while the distributions of different bidders are independent.

(4) To model the setting of correlated item valuations presented in Section 3 we assume that (a) the valuations of a bidder i are given by its *type*, (b) there is a publicly known probability distribution $f_i(t)$ on the types of bidder i with finite support, and (c) $(v_{t1}, \ldots, v_{tm})$ is the vector of valuations for item $1, \ldots, m$ for a bidder with type t. Additionally we assume in this section that every probability $f_i(t)$ is a rational number such that $1/L \leq f_i(t) \leq 1$, where L is polynomial in the size of the input.

3. Correlated item valuations

Here we study the setting in which the distribution of the valuations of a *fixed* bidder for different items can be arbitrarily correlated, while the distributions of different bidders are independent. We present a BIC mechanism that gives a 2-approximation of the optimal revenue.

The mechanism works as follows: Based on the distributions $\mathcal{D}_{ij}$ the mechanism solves a linear programming relaxation of the assignment problem, whose objective function is an upper bound on the value achieved by the optimal mechanism. The linear program returns values for variables $y_{iS}(t)$, where S is an independent set in $\mathcal{M}_i$, for "payment" variables $p_i(t)$ for each i and t, and for variables $x_{ij}(t)$ for each i, j, and t. Then the mechanism interprets y_{iS} as the probability that the optimal BIC mechanism assigns S to i and picks an assignment of items to i based on the probability distribution $y_{iS}(t_i)$, where t_i is the type reported by bidder i. The constraints in the linear program guarantee that the mechanism is BIC.

(LP1) Maximize $\sum_{i,t} f_i(t) p_i(t)$ such that

$$\forall i,j,t \qquad m_{ij} - x_{ij}(t) \geq 0 \tag{1}$$

$$\forall j \qquad -\sum_i m_{ij} \geq -1 \tag{2}$$

$$\forall i,t,s \qquad \sum_j v_{tj} x_{ij}(t) - \sum_j v_{tj} x_{ij}(s) - p_i(t) + p_i(s) \geq 0 \tag{3}$$

$$\forall i,t \qquad \sum_{j \in J_i} v_{tj} x_{ij}(t) - p_i(t) \geq 0 \tag{4}$$

$$\forall i,t,j \qquad \sum_{\text{independent } S \text{ with } j \in S} y_{iS}(t) - x_{ij}(t) = 0 \tag{5}$$

$$\forall i,t \qquad \sum_{\text{independent } S} -y_{iS}(t) \geq -1 \tag{6}$$

$$\forall i,t \qquad -p_i(t) \geq -B_i \tag{7}$$

$$\forall i,j,t : x_{ij}(t) \geq 0, \forall i,t : p_i(t) \geq 0, \forall i,t,S : y_{iS}(t) \geq 0, \forall i,j : m_{ij} \geq 0 \tag{8}$$

We use the auxilliary variable m_{ij}. The reader can see that this does not affect the optimal solution of the LP by letting $m_{ij} := \max_t x_{ij}(t)$. Indeed all inequalities of the LP have to be satisfied. Specifically inequality$-\sum_{i,j} m_{ij} \geq -1$ needs to hold for the input t that maximizes $x_{ij}(t)$ for all i,j, i.e. for t such that $x_{ij}(t) = m_{ij}$ for all i,j.

Note that the optimal BIC mechanism is a feasible solution to LP1: Set $x_{ij}(t)$ to the probability that the mechanism assigns item j to bidder i when the bidder reports type t and $y_{iS}(t)$ to the probability that the mechanism assigns set S to bidder i when the bidder reports type t. This assignment fulfills all constraints of LP1, i.e. it gives a feasible solution to LP1. Thus LP1 has a solution and its optimal solution gives an upper bound on the revenue of the optimal BIC mechanism. LP1 has an exponential number of variables but using the fact that the dual LP has only a polynomial number of variables and a polynomial time separation oracle, we show that LP1 can be solved in polynomial time.

Lemma 1. *The linear program LP1 can be solved in polynomial time.*

Proof of ***Lemma 1****.* Note that LP1 might have an exponential number of variables but it has only a polynomial number of inequalities and equations. Thus the Ellipsoid method with the greedy algorithm as separation oracle can be used to solve the dual of LP1 in polynomial time. More precisely the dual uses the following variables: (a) for each item j a variable g_j, (b) for each bidder i and types t and s a variable b_{its}, (c) for each bidder i and type t variables c_{it}, e_{it}, and h_{it}, and (d) for each bidder i, item j, and type t variables a_{ijt}, d_{ijt} and z_{ijt}. The dual has the following constraints:

(DLP1) Minimize $\sum_j g_j + \sum_{i,t} h_{it} + \sum_{i,t} B_i e_{it}$ such that

$$\forall i,j,t \qquad a_{ijt} - \sum_s v_{tj} b_{its} + \sum_{s'} v_{tj} b_{is't} - v_{tj} c_{it} + z_{ijt} \geq 0 \tag{9}$$

$$\forall i,t \qquad \sum_s b_{its} - \sum_{s'} b_{is't} + c_{it} + e_{it} \geq -f_i(t) \tag{10}$$

$$\forall i,t,\text{independent } S \qquad -\sum_{j \in S} z_{ijt} + h_{it} \geq 0 \tag{11}$$

$$\forall i,j \qquad g_j - \sum_t a_{ijt} \geq 0 \tag{12}$$

$$\forall i,j,t : a_{ijt} \geq 0, \forall i,t,s : b_{its} \geq 0 \tag{13}$$

$$\forall i,t : c_{it} \geq 0, e_{it} \geq 0 \text{ and } h_{it} \geq 0, \forall j : g_j \geq 0 \tag{14}$$

By assumption all values in the constraints are rational numbers and can be turned into integers by multiplying them by L. Additionally setting all variables to 0 gives a feasible solution. Thus, if there exists a polynomial-time separation oracle then the ellipsoid algorithm can solve the dual linear program in time polynomial in the size of the input [14]. Since the Ellipsoid method runs in polynomial time it considers only a polynomial number of inequalities in its computation. Consider a new dual program that contains only these inequalities. It must have the same optimal solution as the original dual. Thus its dual gives rise to new primal program with a polynomial number of variables, whose maximum value is the same as the maximum value of the original primal program. The new primal program can be solved in polynomial time. All variables that are not in the new primal program have value 0.

It remains to give a polynomial-time separation oracle. Note that the greedy algorithm finds in polynomial-time a maximum weight independent set for a matroid. Thus, for every bidder i and every type t it can be used to find a maximum-weight independent set W_i, where the weight of item j is z_{ijt}. If $\sum_{j \in W_i} z_{ijt} \leq h_{it}$ for all i and all t then constraint (12) holds for all i, t and independent sets S. If for some i and t it holds that $\sum_{j \in W_i} z_{ijt} > h_{it}$, we have found a violated constraint. There are only a polynomial number of other constraints and they all can be checked in polynomial time. Thus there exists a polynomial-time separation oracle for the above dual, which leads to a polynomial-time algorithm to solve the above primal program. □

For each bidder i we treat y_{iS} as a probability distribution over the independent sets S of $\mathcal{M}_i$ and pick an independent set T_i according to that probability distribution. We define for all items j, $Z_{0j} = 1$ and for all items j and bidders i let $Z_{ij} = 1 - \sum_{i'<i} x_{i'j}(t_{i'})/2$. Note that $Z_{ij} \geq 1/2$ and thus $1/(2Z_{ij}) \leq 1$. The mechanism assigns the items to bidders as follows:

1. $A = J$
2. For $i = 1, 2, \ldots, n$
   (a) Pick an indep. set T_i using the distribution $y_{S,i}(t_i)$; set $S_i = \emptyset$
   (b) for each $j \in T_i$: if $j \in A$ then with probability $1/(2Z_{ij})$ do:
      i. $S_i = S_i \cup \{j\}$; $A = A - \{j\}$
   (c) Bidder i gets S_i and pays $p_i(t_i)/2$.

We show in the proof of Theorem 1 that $P(j \in S_i) = x_{ij}(t_i)/2$. Thus (3) and (4) of LP1 together with the fact that $p_i = p_i(t_i)/2$ guarantee incentive compatibility in expectation and individual rationality in expectation.

Theorem 1. *The above mechanism is Bayesian incentive compatible, individually rational in expectation, and its revenue is a 2-approximation to the optimal BIC mechanism.*

Proof of ***Theorem 1****.* Due to constraints (1) and (2) in LP1 for each item j $\sum_i x_{ij}(t_i) \leq 1$. Thus $Z_{ij} = 1 - \sum_{i'<i} x_{ij}(t_{i'})/2 \geq 1/2$. Thus, $1/(2Z_{ij}) \leq 1$. During the execution of the mechanism A is the set of items that are still available, while the items in $J \setminus A$ have already been assigned to a bidder. Let E_{ij} be the event that item j belongs to A *after* bidder $i-1$ has been processed (if it exists) and *before* bidder i is processed. We show by induction that (a) $P(E_{ij}) = Z_{ij}$ and (b) $P(j \in S_i) = x_{ij}(t_i)/2$. Note first that, due to Constraint (5) of LP1, for each bidder i item j is in T_i with probability $x_{ij}(t_i)$.

For $i = 0$, the probability that j is in T_i is $x_{0j}(t_0)$, E_{0j} holds and $Z_{0j} = 1$. Thus, $P(E_{0j}) = 1 = Z_{0j}$. The probability that j is in S_i is $P(j \in T_0) \cdot 1/2 = x_{0j}(t_0)/2$.

For $i > 0$ note that $P(E_{ij}) = 1 - P(j \in S_0 \cup j \in S_1 \cup \cdots \cup j \in S_{i-1})$. For all $0 \leq i', i'' < i$, the event $j \in S_{i'}$ is disjoint from the event $j \in S_{i''}$. Thus, $P(j \in S_0 \cup j \in S_1 \cup \cdots \cup j \in S_{i-1})$ $= \sum_{i'<i} P(j \in S_{i'}) = \sum_{i'<i} x_{i'j}(t_{i'})/2$. The latter equation holds by induction. This proves that $P(E_{ij}) = 1 - \sum_{i'<i} x_{i'j}(t_{i'})/2 = Z_{ij}$. Next we analyze $P(j \in S_i)$. The event E_{ij} only depends on the picked sets $T_{i'}$ and the coin flips for bidder i' with $i' < i$, while the event $j \in T_i$ only depends on the randomly chosen set T_i for bidder i. Thus the event E_{ij} is independent from the event $j \in T_i$ and both are independent from the event that the biased coin flip in line (2c) is a success. Thus, $P(j \in S_i) = P(E_{ij}) \cdot P(j \in T_i) \cdot 1/(2Z_{ij})$ $= P(j \in T_j)/2 = x_{ij}(t_i)/2$.

By linearity of expectation, the expected profit bidder i acquires is $\sum_j v_{ij} x_{ij}(t_i)/2 - p_i(t_i)/2 \geq 0$. The latter inequality holds by Constraint (4) in LP1, regardless of the type that bidder i declared. It shows that every bidder is individually rational in expectation. Constraint (3) guarantees that the mechanism is Bayesian incentive compatible.

The expected revenue is $\sum_{i,t} f_i(t) p_i(t)/2$. This is half the value maximized in LP1. Since LP1 is an upper bound to the revenue achieved by the optimal mechanism, our mechanism is a 2-approximation of the optimal revenue. □

4. Independent item valuations

In this section we assume that for each bidder i the distributions of v_{ij} for different j are independent. The goal is to achieve for this case a stronger notion of truthfulness, namely a DSIC instead of a BIC mechanism. All mechanisms in this section are sequential-posted-price mechanisms.

For each item j and bidder i let $\mathcal{V}_{ij} := \min\{v_{ij}, B_i/4\}$ and let f_{ij} be its density function, i.e. $\mathcal{V}_{ij} \sim f_{ij}$. We assume that for all i, j and r all values $f_{ij}(r)$ are rational numbers.

Bhattacharya et al. [3] formulated an LP with variables $x_{ij}(r)$, where $x_{ij}(r)$ denotes the expected amount of item j bidder i gets when $\mathcal{V}_{ij} = r$. We modify their LP by generalizing their constraint for uniform matroids to general matroids and call it LP2.

(LP2) Maximize $\sum_{i,j} \sum_r r f_{ij}(r) x_{ij}(r)$

$$\forall i, \forall S_i \subseteq J \qquad \sum_{j \in S_i} \sum_r f_{ij}(r) x_{ij}(r) \leq \quad \operatorname{rank}(S_i) \tag{15}$$

$$\forall i \qquad \sum_j \sum_r r f_{ij}(r) x_{ij}(r) \leq \quad B_i \tag{16}$$

$$\forall j \qquad \sum_i \sum_r f_{ij}(r) x_{ij}(r) \leq \quad 1 \tag{17}$$

$$\forall i, j, r \qquad x_{ij}(r) \in \quad [0, 1] \tag{18}$$

Remark 1 (**Universal matroid constraints**). *We give here the LPs for the case of* Individual matroid constraints *(the kind of matroid constraints studied in [3]). If there is a* Universal matroid constraint $\mathcal{M}$ *for all players (like the matroid constraints studied in [8]), then the whole set of items allocated to the players has to be an independent set in* $\mathcal{M}$. *For this setting we just need to change the LP inequality that corresponds to the matroid constraint from* $\forall i, \forall S_i \in \mathcal{M}_i$, $\sum_j \sum_r f_{ij}(r) x_{ij}(r) \leq \operatorname{rank}(S_i)$ *to* $\forall S \in \mathcal{M}$, $\sum_{i,j} \sum_r f_{ij}(r) x_{ij}(r) \leq \operatorname{rank}(S)$. *The proof of Lemma 2 applies also to this setting.*

Note that LP2 has an exponential number of constraints, but only polynomially many variables and, as the next lemma shows, it can still be solved in polynomial time.

Lemma 2. *(a) The maximum value of the optimal solution for LP2 achieves at least* $1/4$ *of the revenue of the optimal BIC mechanism. Additionally there exists an optimal solution for LP2 such that* $x_{ij}(r)$ *is a monotonically non-decreasing function of* r. *(b) The solution can be computed in time polynomial in the size of the input.*

*Proof of **Lemma 2**.* (a) The proof that the maximum value is a 4-approximation and that $x_{ij}(r)$ is a monotonically non-decreasing function of r closely follows the proof of [3]. Remember that $\mathcal{V}_{ij} := \min\{v_{ij}, B_i/4\}$ and that f_{ij} is its density function, $x_{ij}(r)$ is the expected amount of item j bidder i gets when $\mathcal{V}_{ij} = r$. Let $p_{ij}(r)$ be the expected price player i pays for item j, if $\mathcal{V}_{ij} = r$ he gets the item. The revenue of the optimal mechanism can be relaxed as follows:

$$\begin{aligned}
&\text{Maximize} \sum_{i,j} \sum_{r} p_{ij}(r) f_{ij}(r) x_{ij}(r) \\
&\forall i, S_i \subseteq J \qquad \sum_{j} \sum_{r} f_{ij}(r) x_{ij}(r) \quad \leq \quad \text{rank}(S_i) \\
&\forall i \qquad \sum_{j \in S_i} \sum_{r} r f_{ij}(r) x_{ij}(r) \quad \leq \quad B_i \\
&\forall j \qquad \sum_{i} \sum_{r} f_{ij}(r) x_{ij}(r) \leq \quad 1 \\
&\qquad \forall i, j, r \qquad x_{ij}(r) \in \quad [0, 1].
\end{aligned}$$

However the previous program is non-linear. We will use the following Claim in order to relax the objective.

Claim 1. *It holds that $p_{ij}(r) \leq 4r$.*

Proof. Case 1: From Individual Rationality we have that $p_{ij}(r) \leq v_{ij}$. If $r = \mathcal{V}_{ij} = v_{ij}$ then obviously $p_{ij}(r) \leq r \leq 4r$. Case 2: If $r = \mathcal{V}_{ij} = B_i/4$ then $B_i = 4r$. Now it has to be $p_{ij}(r) \leq B_i$ and consequently $p_{ij}(r) \leq 4r$. ☐

Now observe that if we scale $p_{ij}(r)$ down by a factor of 4 so that $p_{ij} \leq r$ we preserve the constraints and only loose a factor of 4 in the approximation of the objective. Moreover we can assume without loss of generality that $p_{ij} = r$, because for all i, j with $p_{ij} < r$ we can increase $p_{ij}(r)$ and decrease $x_{ij}(r)$ without changing their product, so that the LP constraints are preserved.

Now the revenue of the optimal mechanism can be approximated by (LP2):

$$\begin{aligned}
&\text{(LP2)} \quad \text{Maximize} \sum_{i,j} \sum_{r} r f_{ij}(r) x_{ij}(r) \\
&\forall i, S_i \subseteq J \qquad \sum_{j} \sum_{r} f_{ij}(r) x_{ij}(r) \quad \leq \quad \text{rank}(S_i) \\
&\forall i \qquad \sum_{j \in S_i} \sum_{r} r f_{ij}(r) x_{ij}(r) \quad \leq \quad B_i \\
&\forall j \qquad \sum_{i} \sum_{r} f_{ij}(r) x_{ij}(r) \leq \quad 1 \\
&\qquad \forall i, j, r \qquad x_{ij}(r) \in \quad [0, 1].
\end{aligned}$$

Finally we can also assume without loss of generality that $x_{ij}(r)$ is monotonically non-decreasing in r. This is because we can decrease $x_{ij}(r)$ for small values of r and increase it for large values of r, so that the objective $\sum_{i,j} \sum_r r f_{ij}(r) x_{ij}(r)$ is preserved, while $\sum_i \sum_r f_{ij}(r) x_{ij}(r)$ decreases and as a result the constraints of the LP are preserved.

(b) To show that the problem can be solved in polynomial time note that all values in the constraints are rational numbers. Additionally, setting all variables to 0 gives a feasible solution. Using the same arguments as in the proof of Lemma 1 we just have to show that we can give a polynomial-time separation oracle. Constraints (17) and (18) require to check a polynomial number of inequalities, each taking polynomial time. We are left to show how to check Constraint (16). For given i and j, let $w_{ij} = \sum_r f_{ij}(r) x_{ij}(r)$. Note that for any set S the function $f(S) = \sum_{j \in S} w_j - rank(S)$ is a supermodular function. A set S that maximizes $f(S)$ can be found in time polynomial in n and m as long as the values w_j are rational numbers as it is identical to minimizing a submodular function (see [19, 21]). The values w_j are rational numbers based on our assumptions on the rationality of the input. Thus, if $f(S)$ is larger than 0, we found a violated constraint. If $f(S)$ is less than 0, then Constraint (16) holds for all subsets S of J. Thus, we can in polynomial time either find a violated constraint or verify that all constraints are fulfilled. □

The following lemma is an important tool that we repeatedly use. It says that by giving the bidder the freedom to choose the independent set he likes, and not assigning him the set with the maximum revenue the mechanism will not lose more than a factor of 2 of the expected revenue.

Lemma 3. *Assume that bidder i is offered a set S of items where each item has the same price r_i. Let q_{ij} be the probability that i is offered item j and picks it. Assume that for every subset T of S, $\sum_{j \in T} q_{ij} \leq rank(T)$. Let S_i be the independent set of $\mathcal{M}_i$ picked by the individually rational bidder i, and let $R_i = \sum_{j \in S_i} q_{ij} r_i$ be the expected revenue from S_i. Then $\sum_{j \in S \setminus S_i} q_{ij} r_i \leq R_i$.*

The equivalent lemma for the case of universal matroid constraints is the following:

Lemma 4. *Assume that each bidder i is offered a set of items where all items have the same price r for all players. Let $S = \cup_i S_i$ be the independent set of the matroid picked by the individually rational bidders, let q_{ij} be the probability that i is offered item j and picks it, and let $R = \sum_i \sum_{j \in S} q_{ij} r$ be the expected revenue from S. If for every subset T of S, $\sum_i \sum_{j \in T} q_{ij} \leq rank(T)$, then $\sum_i \sum_{j \in span(S)} q_{ij} r \leq R$.*

Proof of Lemma 3. Let $span(S_i)$ be the set of elements of the matroid that are blocked by the elements of S_i. Since the bidder is individually rational, he will pick, out of all items that give him a profit, i.e., where $v_{ij} \geq r_i$, a maximal independent set.

Then, as all items are sold at the same price r_i, the expected revenue lost due to the matroid constraints of bidder i, given that the set of elements assigned to i is S_i, is $\sum_{j \in span(S_i)} q_{ij} r_i \leq r_i \sum_{j \in span(S_i)} q_{ij} \leq r_i \operatorname{rank}(span(S_i)) \leq r_i \cdot \operatorname{rank}(S_i)$, where the last term is the revenue obtained. Thus the expected revenue lost is then less than or equal to $\sum_{S_i} r_i \cdot rank(S_i) \cdot P(S_i \text{ is served})$, which is exactly the expected revenue obtained. □

Proof of Lemma 4. Let $span(S)$ be the set of elements of the matroid that are blocked by the elements of S. Since the bidder is individually rational, he will pick a maximal independent set out of all items that give him a profit, i.e., where $v_{ij} \geq r$.

Thus, as all items are sold at the same price r, the expected revenue lost due to the universal matroid constraints, given that the set of elements assigned to i is S_i and $S = \cup_i S_i$, is $\sum_i \sum_{j \in span(S)} q_{ij} r \leq r \sum_i \sum_{j \in span(S)} q_{ij} \leq r \cdot \operatorname{rank}(S)$, where the last term is the revenue obtained. Then the expected revenue lost is then less than or equal to $\sum_S r \cdot rank(S) \cdot P(S \text{ is served})$, which is exactly the expected revenue obtained. □

Using a bucketing technique we show the following result, which does not make any assumptions on the hazard rate.

Theorem 2. *Assume for all i and j, $v_{ij} \in [1, L]$ follow independent distributions f_{ij}. Then there is a $O(\log L)$ approximation of the revenue of the optimal BIC mechanism through a sequential posted price mechanism under any matroid constraint.*

Proof for Individual matroids. The idea for getting a $O(\log L)$ approximation under any matroid constraint is to compute based on the results of LP2 for each bidder i a price r_i^* and for each item j a probability x_{ij}^*. Then the mechanism offers each bidder every remaining item with probability x_{ij}^* at the same price r_i^*. Note that different bidders can set different prices.

To prove Theorem 2 we first solve LP2 to get $x_{ij}(r)$ for all i, j, and r. Then for each bidder i we group the r values in powers of 2 so that there are $\log L$ groups. Let $\mathcal{G}_k$ denote the interval $[2^k, 2^{k+1})$. Let $k_i^* := \text{argmax}_k \sum_j \sum_{r\in\mathcal{G}_k} r f_{ij}(r) x_{ij}(r)$, that is pick the interval that maximizes the expected welfare. We set the price charged to bidder i for each item at $r_i^* := 2^{k_i^*}$ and set $x_{ij}^* = \frac{\sum_{r\in\mathcal{G}^*_{k_i}} f_{ij}(r)x_{ij}(r)}{\sum_{r\in\mathcal{G}^*_{k_i}} f_{ij}(r)}$. Then we use the following mechanism to assign items to bidders.

1. $A = J$.
2. Order the bidders arbitrarily.
3. For $i = 1, 2, \dots, n$
   (a) Let $S = \emptyset$.
   (b) For every item $j \in A$ with probability $x_{ij}^*/2$ add item j to S with a price of r_i^*.
   (c) Let the bidder pick an independent subset S_i of S such that $|S_i| r_i^* \le B_i$. Assign S_i to i at a cost of $|S_i| r_i^*$ and set $A = A \setminus S_i$.

We will now show that the revenue of this mechanism is at least $\Omega(1/\log L)$ of the optimal BIC revenue. Let $q_{ij}^* = P(\mathcal{V}_{ij} \ge r_i^*)$. Let $OPT = \sum_{ij}\sum_r r f_{ij}(r) x_{ij}(r)$ be the optimal solution of LP2. Recall that (by Lemma 2) OPT is at least a constant fraction of the optimal BIC revenue. Thus it suffices to show that the revenue of our mechanism is $\Omega(OPT/\log L)$. Let $OPT_i = \sum_j \sum_r f_{ij}(r) x_{ij}(r)$ be the contribution of bidder i to OPT. We will show that for every bidder i our mechanism receives in expectation $\Omega(OPT_i/\log L)$ from bidder i.

By the second constraint of LP2, $OPT_i \le B_i$. Our mechanism achieves a revenue of $|S_i| r_i^*$ from bidder i. Thus, if $|S_i| r_i^* \ge 3B_i/4$ then our mechanism receives for bidder i a constant fraction of OPT_i. Hence in the following we only need to consider bidders for which $|S_i| r_i^* < 3B_i/4$. Since $\mathcal{V}_{ij} \le B_i/4$, it follows that no item of S was omitted from S_i because of the budget constraint. Said differently, if T_i is the subset of items of S such that $\mathcal{V}_{ij} \ge r_i^*$, then S_i is a maximum rank independent set of T_i.

We next show that:
(1) $r_i^* \sum_j x_{ij}^* q_{ij}^* \ge \dfrac{OPT_i}{2\log L}$ and
(2) our mechanism expects to achieve at least $r_i^* \sum_j x_{ij}^* q_{ij}^*/8$.

We first show (1). (a) By the definition of the buckets and of x_{ij}^* we have

$$\sum_{r\in\mathcal{G}_{k_i{}^*}} r f_{ij}(r) x_{ij}(r) \le 2r_i^* \sum_{r\in\mathcal{G}_{k_i^*}} f_{ij}(r) x_{ij}(r) = 2r_i^* x_{ij}^* \sum_{r\in\mathcal{G}_{k_i^*}} f_{ij}(r) \le 2r_i^* x_{ij}^* q_{ij}^*.$$

By the choice of k_i^* we have that (b) $\sum_j \sum_{r\in\mathcal{G}_{k_i^*}} r f_{ij}(r) x_{ij}(r) \ge \frac{1}{\log L}\sum_j\sum_r r f_{ij}(r) x_{ij}(r)$, as there are totally $\log L$ intervals. Putting (a) and (b) together we get that

$$r_i^* \sum_j x_{ij}^* q_{ij}^* \ge \frac{1}{2\log L}\sum_j\sum_r f_{ij}(r) x_{ij}(r) = \frac{OPT_i}{2\log L}.$$

Now we show (2). Lemma 2 shows that $x_{ij}(r)$ is monotone in r. Thus $x^*_{ij} \leq x_{ij}(r)$ for $r \in \mathcal{G}_k$, with $k > k_i$. Together with the definition of x^*_{ij} this implies (c) $x^*_{ij}q^*_{ij} = \sum_{r \geq r^*_i} f_{ij}(r)x^*_{ij} \leq \sum_r f_{ij}(r)x_{ij}(r)$. Thus by (c) and the third constraint of LP2 we know that (i) for each item j it holds that $\sum_i x^*_{ij}q^*_{ij} \leq 1$. This implies that the probability that j was assigned to a bidder $i' < i$ is upper bounded by $\sum_{i'<i}(x^*_{ij}/2)q^*_{i'j} \leq \frac{1}{2}$. Thus, with probability at least 1/2, j was taken by none of the earlier bidders, which implies that with probability at least 1/2, j is still in A when bidder i is considered.

By (c) and the second constraint of LP2 we know that (ii) for each subset S of J, $\sum_{j \in S} x^*_{ij}q^*_{ij} \leq rank(S)$. Assume for the moment that bidder i had no matroid constraints. Then the expected revenue of bidder i would be

$$R_i := \sum_j P(j \in A) r^*_i \frac{x^*_{ij}}{2} q^*_{ij} \geq \frac{1}{2} \sum_j r^*_i \frac{x^*_{ij}}{2} q^*_{ij}.$$

However, bidder i has to respect the matroid constraints and therefore chooses a maximum independent set of T_i. By (ii) we can use Lemma 3. It shows that that the expected revenue of the items in S_i is at least $R_i/2$. Thus, the expected revenue that our mechanism achieves for bidder i is at least $\sum_j r^*_i x^*_{ij} q^*_{ij}/8$. Together with (1) this shows that the expected revenue that the mechanism achieves for bidder i is at least $OPT_i/(16 \log L)$.

Since the mechanism is a sequential posted price mechanism it is DSIC and individually rational. □

Proof for Global matroids. For one global matroid the mechanism uses one k^* value and one price r^* for all bidders and items. This results in some simplifications in the proof. For completeness we give the full proof.

We first solve LP2 to get $x_{ij}(r)$ for all i, j, and r. Then we group the r values in powers of 2 so that there are $\log L$ groups. Let $\mathcal{G}_k := [2^k, 2^{k+1})$ and $k^* := \text{argmax}_k \sum_{i,j} \sum_{r \in \mathcal{G}_k} r f_{ij}(r) x_{ij}(r)$, that is pick the interval that maximizes the expected welfare. We set the price charged for each item at $r^* := 2^{k^*}$ and set $x^*_{ij} = \frac{\sum_{r \in \mathcal{G}_{k^*}} f_{ij}(r)x_{ij}(r)}{\sum_{r \in \mathcal{G}_{k^*}} f_{ij}(r)}$. Then we use the following mechanism to assign items to bidders.

1. $A = J$.
2. Order the bidders arbitrarily.
3. For $i = 1, 2, \ldots, n$
   (a) Let $G_i = \emptyset$.
   (b) For every item $j \in A$ with probability $x^*_{ij}/3$ add item j to G_i with a price of r^*.
   (c) Let the bidder pick an independent subset S_i of G_i such that $|S_i|r^* \leq B_i$. Assign S_i to i at a cost of $|S_i|r^*$ and set $A = A \setminus \{S_i \cup span(S_i)\}$.

We will now show that the revenue of this mechanism is at least $\Omega(1/\log L)$ of the optimal BIC revenue. Let $OPT = \sum_{i,j} \sum_r r f_{ij}(r) x_{ij}(r)$ be the optimal solution of LP2. Recall that (by Lemma 2) OPT is at least a constant fraction of the optimal BIC revenue. Thus it suffices to show that the revenue of our mechanism is $\Omega(OPT/\log L)$. We will show that our mechanism receives in expectation $\Omega(OPT/\log L)$ from the bidders.

Let $q^*_{ij} = P(\mathcal{V}_{ij} \geq r^*) = \sum_{r \geq r^*} f_{ij}(r)$. Lemma 2 shows that $x_{ij}(r)$ is monotone in r. Thus $x^*_{ij} \leq x_{ij}(r)$ for $r \in \mathcal{G}_k$, with $k > k^*$. Together with the defiition of x^*_{ij} this implies (a) $x^*_{ij}q^*_{ij} = \sum_{r \geq r^*} f_{ij}(r)x^*_{ij} \leq \sum_{r \geq r^*} f_{ij}(r)x_{ij}(r)$.

We show that (1) $r^* \sum_{i,j} x^*_{ij}q^*_{ij} \geq OPT/(2 \log L)$ and (2) our mechanism expects to achieve at least $r^* \sum_{i,j} x^*_{ij}q^*_{ij}/27$.

Proof of (1). By the definition of the buckets and of x_{ij}^* we know that,

$$\sum_{i,j}\sum_{r\in\mathcal{G}_k} rf_{ij}(r)x_{ij}(r) \le 2r^*\sum_{i,j}\sum_{r\in\mathcal{G}_{k^*}} f_{ij}(r)x_{ij}(r) = 2r^*x_{ij}^*\sum_{r\in\mathcal{G}_{k^*}} f_{ij}(r) \le 2r^*x_{ij}^*q_{ij}^*.$$

By the choice of k^* and as there are totally $\log L$ intervals we have that

$$\sum_{i,j}\sum_{r\in\mathcal{G}_{k^*}} rf_{ij}(r)x_{ij}(r) \ge \frac{1}{\log L}\sum_{i,j}\sum_{r} rf_{ij}(r)x_{ij}(r).$$

Combining the two previous inequalities we get that

$$r^*\sum_{i,j} x_{ij}^*q_{ij}^* \ge \frac{1}{2\log L}\sum_{i,j}\sum_{r} f_{ij}(r)x_{ij}(r) = OPT/(2\log L).$$

Proof of (2). By the second constraint of LP2 and (a) we get that for all i,

$$B_i \ge \sum_j\sum_r rf_{ij}(r)x_{ij}(r) \ge r^*\sum_j q_{ij}^*x_{ij}^*.$$

Since $r^*\sum_j q_{ij}^*(x_{ij}^*/3) \le B_i/3$ and from the Markov inequality we get $P(\sum_j r^*q_{ij}^*(x_{ij}^*/3) \ge 3B_i/4) \le 4/9$. Next we show (3). By (a) and the third constraint of LP2 we know that (i) for each item j it holds that $\sum_i x_{ij}^*q_{ij}^* \le 1$. This implies (using the Markov inequality) that the probability that j was assigned to a bidder $i' < i$ is upper bounded by $\sum_{i'<i}(x_{ij}^*/3)q_{i'j}^* \le \frac{1}{3}$.

Thus, by union bounds with probability at least $1-(1/3+4/9) = 2/9$, when the pair (i,j) is considered, j was taken by none of the earlier bidders, and the player also has enough budget to take the item, as $\mathcal{V}_{ij} \le B_i/4$.

Assume for the moment that there are no matroid constraints. Then the expected revenue R would be at least $\sum_j (2/9)r^*(x_{ij}^*/3)q_{ij}^*$. However, the allocation has to respect the matroid constraint and chooses a maximum independent set. By (a) and the second constraint of LP2 we know that for each subset S of J $\sum_i\sum_{j\in S} x_{ij}^*q_{ij}^* \le rank(S)$. We can then use Lemma 4. It shows that that the expected revenue is at least $R/2$. Thus, the expected revenue that our mechanism achieves for bidder i is at least $\sum_{i,j} r_i^*x_{ij}^*q_{ij}^*/27$. Together with (1) this shows that the expected revenue that the mechanism achieves for bidder i is at least $OPT/(54\log L)$.

Since the mechanism is a sequential posted price mechanism it is DSIC and individually rational. □

4.1. Valuation distributions with Monotone Hazard Rate

In this section we give mechanisms with better approximation ratios by placing restrictions on the distribution function f_{ij}. Following Myerson [17] we call the function $h(r) = f(r)/(1-F(r))$ the *hazard rate* of f. The probability distribution f_{ij} has a *monotone hazard rate (MHR)* if $h_{ij}(r)$ is non-decreasing as a function of r. The function $\phi_{ij}(r) = r - 1/h_{ij}(r)$ is called the *virtual valuation* of player i for item j. The distribution is called *regular* if the virtual valuation is a non-decreasing function of r. Clearly, MHR distributions are regular, but the converse does not always hold.

In the previous section we gave a $\Theta(\log L)$ sequential posted price mechanism for general matroids. Since this matches the known lower bound for sequential posted price mechanisms and the l ower bound is achieved by a distribution that satisfies regularity [3], the natural question to ask is whether we can do better for valuations v_{ij} whose distributions satisfy the monotone hazard rate condition.

We modify the LP given by Bhattacharya et al [3] for the special case of uniform, individual matroids to work for both general, individual matroids and general universal matroids. The resulting LP3 has an exponential number of constraints but using the same argument (minimization of submodular function) as in Lemma 2 shows that the LP can be solved in polynomial time. We then generalize the proof of the two subsequent lemmata to work for the modified LPs. We start by giving LP3.

$$
\begin{array}{llll}
\text{(LP3)} & \text{Maximize} \sum_{i,j} \sum_{r} f_{ij}(r)\phi_{ij}(r)x_{ij}(r) & & \\
\forall i, \forall S_i \subseteq J & \sum_{j \in S_i} \sum_{r} f_{ij}(r)x_{ij}(r) \le & \text{rank}(S_i) \\
\forall i & \sum_{j} \sum_{r} f_{ij}(r)\phi_{ij}(r)x_{ij}(r) \le & B_i \\
\forall j & \sum_{i} \sum_{r} f_{ij}(r)x_{ij}(r) \le & 1 \\
\forall i, j, r & x_{ij}(r) \in & [0,1]
\end{array}
$$

The proofs of the following two lemmas follow closely the proofs in [3] and we include them in the Appendix.

Lemma 5. *If the valuations follow distributions that fullfill the MHR condition, then the revenue of LP3 is at least $\frac{1}{2e^2}$ times the revenue of LP2.*

Lemma 6. *The optimal solution to (LP3) satisfies the following property. For all i, j, x_{ij} it can be decomposed in polynomial time in the following way: $x_{ij} = p_{ij}y_{ij} + (1 - p_{ij})z_{ij}$, where $r^*_{ij} + 1 \le \frac{B_i}{4}$, $r^*_{ij} \ge 1$,*

$$
y_{ij}(r) := \begin{cases} 0 & \text{for } r < r^*_{ij} \\ 1 & \text{for } r \ge r^*_{ij} \end{cases} \qquad z_{ij}(r) := \begin{cases} 0 & \text{for } r < r^*_{ij} + 1 \\ 1 & \text{for } r \ge r^*_{ij} + 1. \end{cases}
$$

Define $R_{ij} := p_{ij}r^*_{ij}P(\mathcal{V}_{ij} \ge r^*_{ij}) + (1 - p_{ij})(1 + r^*_{ij})P(\mathcal{V}_{ij} \ge r^*_{ij} + 1)$, and

$$
q_{ij} := p_{ij}P(\mathcal{V}_{ij} \ge r^*_{ij}) + (1 - p_{ij})P(\mathcal{V}_{ij} \ge r^*_{ij} + 1).
$$

Then (a) $\sum_r x_{ij}(r)\phi_{ij}(r)f_{ij}(r) = R_{ij}$, and (b) $\sum_r x_{ij}(r)f_{ij}(r) = q_{ij}$.

4.1.1. An $O(k)$ Approximation for General Matroids assuming MHR

As a warmup note that the following very simple mechanism achieves an $O(k)$ approximation ratio.

Theorem 3. *Assume for all i and j, $v_{ij} \in [1, L]$ follow independent distributions f_{ij} and satisfy the monotone hazard rate condition. Then there is an $O(k)$ approximation of the revenue of the optimal BIC mechanism, through a spp mechanism under any matroid constraint. Here k is the rank of the matroid for the case of global matroids and $\max_{1 \le i \le n} k_i$ where k_i is the rank of the matroid constraint of player i for individual matroids.*

We first solve LP3 to get the x_{ij} values and then decompose $\tilde{x}_{ij} = x_{ij}/(2k)$ according to Lemma 6 to get p_{ij}, r^*_{ij} and q_{ij} values.

Let $\tilde{r}_{ij}$ be the random variable that is equal to r^*_{ij} with probability p_{ij} and equal to $r^*_{ij}+1$ with probability $1-p_{ij}$. The mechanism is the following:

1. $A = J$.
2. Order the bidders arbitrarily.
3. For $i = 1, 2, \ldots, n$
   (a) For every item $j \in A$ post for it a price of $\tilde{r}_{ij}$.
   (b) Ask bidder i to select at most one item j from A without violating his matroid and budget constraints. Set $A = A \setminus \{j\}$.

(In the case of one global matroid in step (b) the algorithm should stop when the first item is selected by a bidder, so we set $A = \emptyset$ as soon as the first item is allocated.) We include the proof of Theorem 3 in the appendix.

4.1.2. An $O(\log m)$ approximation for general matroids assuming MHR

To achieve an $O(\log m)$ approximation we bucket for each bidder i all the items above a certain threshold according to their r^*_{ij} value. Note that this is different from the bucketing in the proof of Theorem 2, where we placed the r values into buckets. Then we pick the bucket with the largest expected revenue, assign all items in it the same price, and let the bidder pick an independent set of items from the bucket. We use Lemma 3 to show that the expected revenue lost due to the matroid constraints is only a factor of 2.

Mechanism for Individual matroid constraints

1. Solve LP3 to get the $x_{ij}(r)$ values, set for all i, j and r, $\tilde{x}_{ij}(r) = x_{ij}(r)/2$. Decompose $\tilde{x}_{ij}(r)$ to get p_{ij} and r^*_{ij} values for all i and j. Set $R_{ij} = p_{ij} r^*_{ij} P(\mathcal{V}_{ij} \geq r^*_{ij}) + (1 - p_{ij})(1 + r^*_{ij}) P(\mathcal{V}_{ij} \geq r^*_{ij} + 1)$.
2. For each bidder i do
   (a) Set $OPT_i = \sum_j R_{ij}$, let $r_i^{max} = \max_j\{r^*_{ij}+1\}$, and let $r_i^{min} = OPT_i/m^2$.
   (b) For $l = \lfloor \log(r_i^{min}) \rfloor$ to $\lfloor \log(r_i^{max}) \rfloor$ do: Set $\Gamma_l = \emptyset$
   (c) For all items j with $r^*_{ij} \geq r_{min}$ do:
      i. Set $k = \lfloor \log r^*_{ij} \rfloor$, $\Gamma_k = \Gamma_k \cup \{(j, p_{ij} \cdot P(\mathcal{V}_{ij} \geq r^*_{ij})/P(\mathcal{V}_{ij} \geq 2^k))\}$, set $k' = \lfloor \log(r^*_{ij}+1) \rfloor$ and $\Gamma_{k'} = \Gamma_{k'} \cup \{(j, (1-p_{ij}) \cdot P(\mathcal{V}_{ij} \geq r^*_{ij}+1)/P(\mathcal{V}_{ij} \geq 2^{k'}))\}$
   (d) Set $\mathcal{B}_i := \Gamma_{k_i}$ with $k_i = \text{argmax}_k \sum_{(j,p)\in\Gamma_k} p 2^k P(\mathcal{V}_{ij} \geq 2^k)$.
3. $A = J$ and order the bidders arbitrarily.
4. For $i = 1, 2, \ldots, n$
   (a) Let $S = \emptyset$.
   (b) For every item $j \in A$: If $(j,p) \in \mathcal{B}_i$ then with probability p add item j to S with a price of 2^{k_i}.
   (c) Let the bidder pick an independent subset S_i of S such that $|S_i| 2^{k_i} \leq B_i$. Assign S_i to i at a cost of $|S_i| 2^{k_i}$ and set $A = A \setminus S_j$.

Theorem 4. *Suppose that the valuations $v_{ij} \in [1, L]$ follow independent distributions f_{ij} for different (i,j) satisfying the monotone hazard rate condition and that the items allocated are subject to a matroid constraint. Then for every bidder i the above mechanism is an $O(\log(r_i^{max}/r_i^{min}))$ approximation of OPT_i under any individual matroid constraint.*

Proof for Individual matroids. Due to division of x_{ij} by 2 we have that $\sum_i OPT_i$ is at least half of the revenue of LP3. We will show that our mechanism receives a revenue of at least $(\sum_i OPT_i)/(8\lfloor\log(r_i^{max}/r_i^{min})\rfloor)$, which shows that the mechanism is a $O(\log(r_i^{max}/r_i^{min}))$ approximation of the optimal BIC mechanism.

Fix a bidder i. The total revenue discarded by ignoring all items j with $r_{ij} < r_i^{min} = OPT_i/m^2$ is at most $OPT_i/m \leq OPT_i/2$. Thus $\sum_{j \text{ with } r_{ij} \geq r_i^{min}} R_{ij} \geq OPT_i/2$. We will show that (A) the above mechanism collects a revenue of at least a $1/(16\lfloor\log(r_i^{max}/r_i^{min})\rfloor)$ fraction of $\sum_{j \text{ with } r_{ij} \geq r_i^{min}} R_{ij}$.

Note that $OPT_i \leq B_i$. Our mechanism collects from bidder i the revenue $|S_i|2^{k_i}$. If $|S_i|2^{k_i} \geq 3B_i/4$, our mechanism received a constant fraction of OPT_i, which shows (A). Thus, we only need to analyze bidders i where $|S_i|2^{k_i} < 3B_i/4$. Since $2^{k_i} \leq r^*_{ij} + 1 \leq B_i/4$ no item of S was rejected by such bidders i because of the budget constraint. This implies that the subset S_i picked by such bidders i is a maximum independent subset of T_i, the subset of all items j of S with $\mathcal{V}_{ij} \geq 2^{k_i}$. We first need to show the following three claims.

Claim 2. $\sum_{(j,p)\in\mathcal{B}_i} p2^{k_i} P(\mathcal{V}_{ij} \geq 2^{k_i}) \geq OPT_i/(4\lfloor\log(r_i^{max}/r_i^{min})\rfloor)$.

Proof. Recall that only items with $r_{ij} \geq r_i^{min}$ are placed into buckets. Thus

$$\sum_{k=\lfloor\log(r_i^{\min})\rfloor}^{\lfloor\log(r_i^{\max})\rfloor} \sum_{(j,p)\in\Gamma_k} p2^k \; P(\mathcal{V}_{ij} \geq 2^k)$$

$$\geq \sum_{j \text{ with } r^*_{ij} \geq r_i^{min}} p_{ij} \frac{P(\mathcal{V}_{ij} \geq r^*_{ij})}{P(\mathcal{V}_{ij} \geq 2^{\lfloor\log r^*_{ij}\rfloor})} \cdot \frac{r^*_{ij}}{2} \cdot P(\mathcal{V}_{ij} \geq 2^{\lfloor\log r^*_{ij}\rfloor}) +$$

$$(1-p_{ij}) \cdot \frac{P(\mathcal{V}_{ij} \geq r^*_{ij}+1)}{P(\mathcal{V}_{ij} \geq 2^{\lfloor\log(r^*_{ij}+1)\rfloor})} \cdot \frac{r^*_{ij}+1}{2} \cdot P(\mathcal{V}_{ij} \geq 2^{\lfloor\log(r^*_{ij}+1)\rfloor})$$

$$= \sum_{j \text{ with } r_{ij} \geq r_i^{min}} R_{ij}/2 \geq OPT_i/4.$$

The claim follows by the fact that there are at most $\lfloor\log(r_i^{max}/r_i^{min})\rfloor$ many buckets and by the choice of $\mathcal{B}_i$. □

Claim 3. *For every bidder i, let A_i be the set A when the processing of bidder i starts. Then for every item j, it holds that j belongs to A_i with probability at least $1/2$.*

Proof. Item j belong to A_i iff none of the bidders $i' < i$ selected j. The probability that bidder i' selected j is at most $p_{i'j}P(\mathcal{V}_{i'j} \geq r^*_{i'j}) + (1-p_{i'j})P(\mathcal{V}_{i'j} \geq r^*_{i'j}+1) = q_{i'j} = \sum_r \tilde{x}_{i'j}(r)f(r) = \sum_r x_{i'j}(r)f(r)/2$. By constraint three of LP3 we have that $\sum_{i',i}\sum_r x_{i'j}(r)f(r)/2 < 1/2$. Thus the probability that any earlier bidder selected j is at most $1/2$. □

Claim 4. *For every bidder i and subset S of J, it holds that* $\sum_{j\in S \text{ and } (j,p)\in\mathcal{B}_i} pP(\mathcal{V}_{ij} \geq 2^{k_i}) \leq rank(S)$.

Proof. Note that by the first constraint of LP3

$$\sum_{j\in S \text{ and } (j,p)\in\mathcal{B}_i} pP(\mathcal{V}_{ij} \geq 2^{k_i}) \leq \sum_{j\in S} q_{ij} \leq rank(S).$$

□

Assume for the moment that bidder i did not have any matroid constraints. Then by the second claim the expected revenue of bidder i would be $X := \sum_{(j,p)\in\mathcal{B}_i} P(j \in A) p 2^{k_i} P(\mathcal{V}_{ij} \geq 2^{k_i}) \geq \sum_{(j,p)\in\mathcal{B}_i} p 2^{k_i} P(\mathcal{V}_{ij} \geq 2^{k_i})/2$. Thus by the first claim $X \geq OPT_i/(8\lfloor \log(r_i^{max}/r_i^{min})\rfloor)$. Note that bidder i is offered all items in $\mathcal{B}_i$ for the same price and picks a maximum independent subset of T_i. Additionally for every subset S of J it holds that $\sum_{j\in S \text{ and } (j,p)\in\mathcal{B}_i} pP(\mathcal{V}_{ij} \geq 2^{k_i}) \leq rank(S)$ by the third claim. Thus, by Lemma 3, the expected revenue of bidder i is at least $X/2$. It follows that the expected revenue of bidder i is at least $OPT_i/(16\lfloor \log{(r_i^{max}/r_i^{min})}\rfloor)$. □

Corollary 1. *If $f_{ij}(r) \geq 1/m^c$ for all i, j, and r and some constant c, then our mechanism is an $O(\log m)$ approximation of the optimal revenue.*

of Corollary 1. Recall that $r_i^{min} = OPT_i/m^2$. If we assume that $f_{ij}(r) \geq 1/m^c$ for all i, j, and r and some constant c, then it follows that $P(\mathcal{V}_{ij} \geq r^*_{ij} + 1) \geq 1/m^c$. Since we know that for all i and j $r^*_{ij} \cdot P(\mathcal{V}_{ij} \geq r^*_{ij} + 1) \leq R_{ij} \leq OPT_i$ and $r^*_{ij} \geq 1$, it follows that

$$r_i^{max} \leq \max_j r^*_{ij} + 1 \leq 2 \max_j r^*_{ij} \leq 2m^c OPT_i \leq m^{c+1} OPT_i.$$

Thus under this assumption $r_i^{max}/r_i^{min} \leq m^{c+3}$ and, hence, our mechanism is an $O(\log m)$ approximation of the optimal revenue. □

Mechanism for Global matroid constraints

1. Solve LP3 to get the $x_{ij}(r)$ values and set for all i, j and r, $\tilde{x}_{ij}(r) = x_{ij}(r)/3$. Decompose $\tilde{x}_{ij}(r)$ to get p_{ij} and r^*_{ij} values for all i and j. Set $R_{ij} = p_{ij} r^*_{ij} P(\mathcal{V}_{ij} \geq r^*_{ij}) + (1-p_{ij})(1+r^*_{ij}) P(\mathcal{V}_{ij} \geq r^*_{ij}+1)$, let $OPT = \sum_{i,j} R_{ij}$, let $r_{max} = \max_{ij}\{r^*_{ij}+1\}$, and let $r_{min} = OPT/m^2$.
2. For each bidder i do
   (a) For $l = \lfloor \log(r_{min})\rfloor$ to $\lfloor \log(r_{max})\rfloor$ do: Set $\Gamma_l = \emptyset$
   (b) For all items j with $r^*_{ij} \geq r_{min}$ do:
      i. Set $k = \lfloor \log r^*_{ij}\rfloor$, $\Gamma_k = \Gamma_k \cup \{(j, p_{ij} \cdot P(\mathcal{V}_{ij} \geq r^*_{ij})/P(\mathcal{V}_{ij} \geq 2^k))\}$, set $k' = \lfloor \log(r^*_{ij}+1)\rfloor$ and $\Gamma_{k'} = \Gamma_{k'} \cup \{(j, (1-p_{ij}) \cdot P(\mathcal{V}_{ij} \geq r^*_{ij}+1)/P(\mathcal{V}_{ij} \geq 2^{k'}))\}$
3. Find the bucket $\mathcal{B} := \Gamma_{k^*}$ with $k^* = \text{argmax}_k \sum_i \sum_{(j,p)\in\Gamma_k} p 2^k P(\mathcal{V}_{ij} \geq 2^k)$.
4. $A = J$.
5. Order the bidders arbitrarily.
6. For $i = 1, 2, \ldots, n$
   (a) Let $S = \emptyset$.
   (b) For every item $j \in A$: If $(j,p) \in \mathcal{B}$ then with probability p add item j to S with a price of 2^{k^*}.
   (c) Let the bidder pick an independent subset S_i of S such that $|S_i| 2^{k^*} \leq B_i$. Assign S_i to i at a cost of $|S_i|2^{k^*}$ and set $A = A \setminus \{S_i \cup span(S_i)\}$.

The main technical challenge for proving the same result as in Theorem 4 for global matroid constraints is that the argument cannot be applied to each bidder individually and thus the argument that deals with budget constraints cannot be used. Thus in the mechanism we need to divide $x_{ij}(r)$ by 3, which allows us to show that with constant probability the budget constraint of bidder i is not violated.

Theorem 5. *Suppose that the valuations $v_{ij} \in [1, L]$ follow independent distributions f_{ij} for different (i,j) satisfying the monotone hazard rate condition and that the items allocated* *are subject to a matroid constraint. Then the above mechanism is an* $O(\log{(r_{max}/r_{min})})$ *approximation of the optimal BIC revenue under any global matroid constraint.*

Proof for Global matroids. Due to division of x_{ij} by 3 we have that OPT is at least 1/3 of the revenue of the optimal BIC mechanism. We will show that our mechanism receives a revenue of at least $(OPT)/(2 \cdot 4 \cdot 9\lfloor\log{(r_{max}/r_{min})}\rfloor)$, which shows that the mechanism is a $O(\log{(r_{max}/r_{min})})$ approximation of the optimal BIC mechanism.

The total revenue discarded by ignoring all items j with $r_{ij} < r_{min} = OPT/m^2$ is at most $OPT/m \leq OPT/2$. Thus $\sum_i \sum_{j \text{ with } r_{ij} \geq r_{min}} R_{ij} \geq OPT/2$ and consequently by ignoring all items with $r_{ij} < r_{min}$ we loose at most a factor of 2 in the approximation of the optimal revenue.

Claim 5. $\sum_i \sum_{(j,p)\in\mathcal{B}} p2^k P(\mathcal{V}_{ij} \geq 2^k) \geq OPT/(4\lfloor\log(r_{max}/r_{min})\rfloor)$.

Proof. Recall that only items with $r_{ij} \geq r_{min}$ are placed into buckets.

$$\sum_{k=\lfloor log(r_{\min})\rfloor}^{\lfloor\log(r_{\max})\rfloor} \sum_i \sum_{(j,p)\in\Gamma_k} p2^k\ P(\mathcal{V}_{ij} \geq 2^k)$$
$$\geq \sum_i \sum_{j \text{ with } r_{ij}\geq r_{min}} p_{ij}P(\mathcal{V}_{ij} \geq r^*_{ij})/P(\mathcal{V}_{ij} \geq 2^{\lfloor\log r^*_{ij}\rfloor})(r^*_{ij}/2)P(\mathcal{V}_{ij} \geq 2^{\lfloor\log r^*_{ij}\rfloor})+$$
$$(1-p_{ij})P(\mathcal{V}_{ij} \geq r^*_{ij}+1)/P(\mathcal{V}_{ij} \geq 2^{\lfloor\log r^*_{ij}+1\rfloor})((r^*_{ij}+1)/2)P(\mathcal{V}_{ij} \geq 2^{\lfloor\log{(r^*_{ij}+1)}\rfloor})$$
$$= \sum_i \sum_{j \text{ with } r_{ij}\geq r_{min}} R_{ij}/2 \geq OPT/4.$$

The claim follows by the fact that there are $\lfloor\log(r_{max}/r_{min})\rfloor$ many buckets and the choice of $\mathcal{B}$. □

Claim 6. *For every bidder i and item j, it holds that none of the bidders $i' < i$ selected j with probability at least* $1/3$.

Proof. The probability that bidder i' selected j is at most $p_{i'j}P(\mathcal{V}_{i'j} \geq r^*_{i'j}) + (1-p_{i'j})P(\mathcal{V}_{i'j} \geq r^*_{i'j}+1) = q_{i'j} = \sum_r \tilde{x}_{i'j}(r)f(r) = \sum_r x_{i'j}(r)f(r)/3$. By constraint three of LP3 we have that $\sum_{i'}\sum_r x_{i'j}(r)f(r)/3 < 1/3$. Thus the probability that any earlier bidder selected j is at most 1/3. □

Claim 7. *For every bidder i and subset S of J it holds that*
$\sum_{j\in S \text{ and } (j,p)\in\mathcal{B}} pP(\mathcal{V}_{ij} \geq 2^{k^*}) \leq rank(S)$.

Proof. Note that $\sum_i \sum_{j\in S \text{ and } (j,p)\in\mathcal{B}} pP(\mathcal{V}_{ij} \geq 2^{k^*}) \leq \sum_i \sum_{j\in S} q_{ij} \leq rank(S)$ by the first constraint of LP3. □

Claim 8. *For every bidder i and subset S of J it holds that*
$\sum_{j\in S \text{ and } (j,p)\in\mathcal{B}} p2^{k^*}P(\mathcal{V}_{ij} \geq 2^{k^*}) \leq B_i/3$.

Proof. From the LP constraints we have

$$\sum_{j\in S \text{ and } (j,p)\in\mathcal{B}} p2^{k^*}P(\mathcal{V}_{ij} \leq 2^{k^*}) \leq \sum_j R_{ij} = \sum_j \sum_r f_{ij}(r)\phi_{ij}(r)x_{ij}(r)/3 \leq B_i/3.$$

□

Assume for the moment that bidder i did not have any matroid constraints. Let P_{ij} denote the revenue extracted from the pair (i, j), (which is zero if item j is not sold to player i) and $X_{ij} \in \{0, 1\}$ the random variable denoting whether item i was taken by player j. Let E_1 denote the event $\{\sum_{l \neq j} X_{il} \geq 1\}$ and E_2 the event $\{\sum_{k \neq j} P_{ik} \geq 3B_i/4\}$.

By the previous two claims, applying the Markov inequality, we get $P(E_1) \leq 1/3$ and $P(E_2) \leq 4/9$. Thus we have $P(E_1 \cup E_2) \leq P(E_1) + P(E_2) = 7/9$ and consequently with probability at least $2/9$ none of these two events occurs. So when the pair (i, j) is considered, with probability at least $2/9$, j was taken by none of the earlier bidders, and the player also has enough budget to take the item, as $\mathcal{V}_{ij} \leq B_i/4$. Then if we denote the expected revenue by X, it would be $X \geq \sum_{(j,p) \in \mathcal{B}} 2/9p2^{k^*} P(\mathcal{V}_{ij} \geq 2^{k^*})$. Thus by the first claim $X \geq 2 \cdot OPT/(4 \cdot 9\lfloor \log(r_{max}/r_{min})\rfloor)$. Note that bidder i is offered all items in $\mathcal{B}$ for the same price and picks a maximum independent subset of T_i. Additionally for every subset S of J it holds that $\sum_{j \in S \text{ and } (j,p) \in \mathcal{B}} pP(\mathcal{V}_{ij} \geq 2^{k^*}) \leq rank(S)$ by the third claim. Thus, by Lemma 3, we loose at most another factor of 2 in the expected revenue. It follows that the expected revenue is at least $OPT/(4 \cdot 9\lfloor \log(r_{max}/r_{min})\rfloor)$. ☐

Thus using the same proof as for Corollary 1 with OPT_i replaced by OPT we get the following result.

Corollary 2. *If we assume that $f_{ij}(r) \geq 1/m^c$ for all i, j, and r and some constant c, then our mechanism is an $O(\log m)$ approximation of the optimal revenue.*

4.2. Constant approximations for specific matroids with the MHR assumption

If the valuations v_{ij} follow MHR distributions and the feasibility constraint is described by a k-uniform matroid, [3] gives a constant approximation algorithm for the optimal expected revenue. We improved the approximation ratio from 24 [3] to 9, by arguing differently (and more simply) about the fulfillment of the budget constraints: *If the expected revenue from a bidder i is at least $3/4B_i$, the mechanism achieved a constant factor of the optimal revenue for i. Otherwise, since $\mathcal{V}_{ij} \leq B_i/4$, the budget constraints did not keep i from taking more items and can be ignored in the future analysis of the revenue collected from i.* The proof also easily extends for the case of a universal matroid constraint.

Theorem 6. *Assume that for all i and j, $v_{ij} \in [1, L]$ follow independent distributions f_{ij} and satisfy the monotone hazard rate condition. There is a constant approximation of the revenue of the optimal BIC mechanism, that achieves a 9-approximation to LP3, through a spp mechanism under global or individual k-uniform matroid constraints.*

Proof for individual and global matroids. We first solve LP3 to get the x_{ij} values and then decompose $x_{ij}/3$ according to Lemma 6 to get p_{ij}, r^*_{ij} and q_{ij}.

1. $A = J$.
2. Order the bidders arbitrarily.
3. For $i = 1, 2, \ldots, n$
   (a) For every item $j \in A$ add it to the set of offered items with probability $x_{ij}/3$ and post for it a price of $\tilde{r}_{ij}$.
   (b) Ask bidder i to pick the items he prefers (without violating his budget and matroid constraints). Denote by S_i the set of elements picked by bidder i and set $A = A \setminus S_i$.

Here we use our new technique for dealing with budgets. The idea is that either the revenue of the mechanism is a constant fraction of the budget, so close to the best possible revenue achievable, or the budget did not force the player to reject any of the items he was offered.

Let P_{ij} denote the revenue extracted from the pair (i, j), (which is zero if item j is not sold to player i) and $X_{ij} \in \{0, 1\}$ the random variable denoting whether item i was taken by player j. Note that $OPT_i \leq B_i$. Our mechanism collects from bidder i the revenue $\sum_j P_{ij}$.

We take two cases: (a) If $\sum_j P_{ij} \geq 3B_i/4$, then $\sum_j P_{ij} \geq 3B_i/4 \geq 3OPT_i/4$ so our mechanism receives a $3/4$ fraction of OPT_i and the proof is complete.

(b) If $\sum_j P_{ij} < 3B_i/4$, the following argument shows that the budget of player i did not affect his decision to take or leave any of the items since $\tilde{r}_{ij} \leq r^*_{ij} + 1 \leq B_i/4$ for all j.

(i) (***Individual matroids***) Fix a player i and an item j. We will show that with constant probability item j is still available and the total number of items k that can be allocated has not been exceeded yet, so the player picks the item if $v_{ij} \geq \tilde{r}_{ij}$. Let E_1 denote the event $\{\sum_{s \neq i} X_{sj} \geq 1\}$ and E_2 the event $\{\sum_{l \neq j} X_{il} \geq k\}$.

By Lemma 6 and the first and third constraint of LP3 it follows that

$$E[\sum_{j \neq j} X_{il}] = \sum_l q_{il} = \sum_l \sum_r (x_{il}(r)/3) f_{il}(r) \leq k/3 \textit{ and } E[\sum_{s \neq i} X_{sj}] \leq 1/3.$$

By applying the Markov inequality we get $P(E_1) \leq 1/3$ and $P(E_2) \leq 1/3$. Thus, we have $P(E_1 \cup E_2) \leq P(E_1) + P(E_2) = 2/3$ and consequently with probability at least $1/3$ none of these two events occurs. Consequently for each pair (i, j) we obtain a $1/3$ approximation of the revenue, thus for each player i we obtain a $1/3$ approximation of the OPT_i.

Putting cases (a) and (b) together we get at least a $1/3 \cdot \min\{1/3, 3/4\} = 1/9$ approximation of the objective of LP3.

(ii) (***Global matroid***) Fix a player i and an item j. We will show that with constant probability item j is still available and the total number of items s that can be allocated has not been exceeded yet, so the player picks the item if $v_{ij} \geq \tilde{r}_{ij}$. Let E_1 denote the event $\{\sum_{l \neq j} X_{il} \geq 1\}$ and E_2 the event $\{\sum_{s \neq i} \sum_{l \neq j} X_{sl} \geq k\}$.

By Lemma 6 and the first and third constraint of LP3 it follows that

$$\sum_{i,j} q_{ij} = \sum_{i,j} \sum_r (x_{ij}(r)/3) f_{ij}(r) \leq k/3 \quad \text{and } E[\sum_{s \neq i} X_{sj}] \leq 1/3.$$

By applying the Markov inequality we get $P(E_1) \leq 1/3$ and $P(E_2) \leq 1/3$. Thus we have $P(E_1 \cup E_2) \leq P(E_1) + P(E_2) = 2/3$ and consequently with probability at least $1/3$ none of these two events occurs. Consequently for each pair (i, j) we obtain a $1/3$ approximation of the revenue, thus for each player i we obtain a $1/3$ approximation of the OPT_i.

Putting cases (a) and (b) together we get at least a $1/3 \cdot \min\{1/3, 3/4\} = 1/9$ approximation of the objective of LP3. □

4.3. Graphical matroids

In the case of graphical matroids we use a graph decomposition technique employed by [8] for matroids with budget constraints and prophet inequalities for matroids without budget constraints to achieve the following results.

Theorem 7. *If for all i and j, $v_{ij} \in [1, L]$ follow independent distributions f_{ij} and satisfy the monotone hazard rate condition, then there is a constant approximation of the revenue of LP3, through a sequential posted price mechanism under global or individual graphical matroid constraints.*

Remark 2. *Our new technique for dealing with budgets applies to all cases where we can argue that we achieve a constant fraction of the optimal revenue R_{ij} for each bidder item pair (i, j) (like for example in the case of Uniform matroids) or of the optimal revenue $OPT_i = \sum_j R_{ij}$ for each player separately (like in the case of any global matroid constraint). In those cases we loose at most a factor of 3/4 because of the budget constraint. If however we can only argue about the total revenue for all items (like in graphical matroids) then budget constrained bidders together with global matroids result in much worse approximation ratios than for the case of individual matroids together with budget constraints.*

Proof of ***Theorem 7 for individual matroids***. We first solve LP3 to get the x_{ij} values and then decompose $x_{ij}/4$ according to Lemma 6 to get p_{ij}, r^*_{ij} and q_{ij}. By Lemma 6 and the first constraint of LP1 it follows that $\sum_j q_{ij} = \sum_{j\in S}\sum_r (x_{ij}(r)/4) f_{ij}(r) \leq \text{rank}(S)/4$ for each i and each subset S of J.

Let $G_i = (V_i, J)$ be the graph defining the feasibility constraint for bidder i. The idea is to partition the matroid into sets $T_{i,v}$ so that bidder i gets only a single item from each $T_{i,v}$ and still obtain a good approximation to the expected optimal revenue. We can achieve this if we can find a partition of the items (which are the edges of G_i), into sets $T_{i,v}$ such that for each i it holds that $\cup_{v\in V_i} T_{i,v} = J$ and $\sum_{j\in T_{i,v}} q_{ij} \leq 1/2$.

We can find such a partition as follows: For each $v \in V_i$ let $\delta_i(v)$ be the set of edges incident to v and define $q^i_v := \sum_{j\in\delta_i(v)} q_{ij}$. Now $\sum_v q^i_v = 2\sum_j q_{ij} \leq 2\cdot\text{rank}(V_i)/4 \leq (|V_i| - 1)/2$. Thus there exists a v for which $q^i_v \leq 1/2$. For each bidder i we run the following algorithm independently.

1. $U = V_i$, $A = J$. While there exists a $v \in U$
   (a) For every node v set $q^i_v = \sum_{j\in\delta_i(v)\cap A} q_{ij}$
   (b) Pick a node $v \in U$ with $q^i_v \leq 1/2$,
   (c) Set $T_{i,v} := \delta_i(v) \cap A$,
   (d) Set $U = U \setminus \{v\}$ and $A = A \setminus T_{i,v}$.

Fix a bidder i. The sets $T_{i,v}$ are chosen in such a way that if a set S_i contains no more than one element from $T_{i,v}$ for each v, then S_i is an independent set in $\mathcal{M}_i$, i.e., the matroid constraints are fulfilled. Now that we have the desired partition we can present our algorithm. Let $\tilde{r}_{ij}$ be the random variable that is equal to r^*_{ij} with probability p_{ij} and equal to $r^*_{ij} + 1$ with probability $1 - p_{ij}$.

1. $A = J$.
2. Order the bidders arbitrarily.
3. For $i = 1, 2, \ldots, n$
   (a) For every item $j \in A$ post for it a price of $\tilde{r}_{ij}$.
   (b) Ask bidder i to pick the items he prefers (without violating his budget constraints) with the restriction that he can only take a single item from each set $T_{i,v} \cap A$. Denote by S_i the set of elements picked by bidder i and set $A = A \setminus S_i$.

Let $OPT_i = \sum_j R_{ij}$. We will show that the revenue achieved by our mechanism for every bidder i is a constant factor of OPT_i. Note that $OPT_i \leq B_i$. Thus if the mechanism achieves a revenue of $3B_i/4$ for bidder i it has achieved a constant factor of OPT_i. Thus in the following we only consider bidders i where our mechanism achieves a revenue of less than $3B_i/4$. In this case bidder i is not constrained by its budget since for every item j it holds that $\tilde{r}_{ij} \leq B_i/4$. Thus if $A \cap T_{i,v} \neq \emptyset$ when bidder i is considered, then bidder i will pick an element j of $A \cap T_{i,v}$, provided $v_{ij} \geq \tilde{r}_{ij}$. Thus we will show that our mechanism obtains in each one of these sets $T_{i,v}$ a constant approximation of the expected revenue this $T_{i,v}$ contributes to the optimal solution. Fix a set $T_{i,v}$ and an item $j \in T_{i,v}$. If $j \in T_{i,v} \cap A$ and $T_{i,v} \cap A$ doesn't contain any other item that he prefers to item j, then we are sure that the expected revenue extracted is R_{ij}. Let C_1 denote the event $\{j \in A$ when bidder i is considered$\}$ and C_2 denote the event $\{$bidder i selects no item that is not j from $T_{i,v} \cap A\}$. Then the expected revenue our algorithm extracts from each $T_{i,v}$ is at least $\sum_{j \in T_{i,v}} P(C_1 \cap C_2) \cdot R_{ij} = \sum_{j \in T_{i,v}} P(C_1) \cdot P(C_2 \mid C_1) \cdot R_{ij}$ and the total expected revenue of the algorithm is at least $\sum_i \sum_v \sum_{j \in T_{i,v}} P(C_1) \cdot P(C_2 \mid C_1) \cdot R_{ij} = \sum_i \sum_j P(C_1) \cdot P(C_2 \mid C_1) \cdot R_{ij}$.

It remains to show that $P(C_1) \geq 1/2$ and $P(C_2 \mid C_1) \geq 1/2$. The probability $P(\{j \in A$ when bidder i is considered$\})$, i.e., the probability that j has not been allocated to any of the previous bidders, is at least $3/4$, because from Lemma 6 and the constraints of LP3 we have that for each j, $E[\sum_{i'<i} q_{i'j}] \leq E[\sum_i \sum_r f_{ij}(r) x_{ij}(r)/4] \leq 1/4$ and consequently by the Markov inequality $P[\sum_{i'<i} q_{i'j} \geq 1] \leq 1/4$, i.e., $P[\sum_{i'<i} q_{i'j} < 1] \geq 3/4$.

It holds that

$$\begin{aligned} P(C_2 \mid C_1) &= P(\{i \text{ picks no other item from } T_{i,v} \cap A \mid C_1\}) \\ &\geq P(\{i \text{ has a value below the price for all items of } (T_{i,v} \cap A) \setminus \{j\}\} \mid C_1) \\ &\geq P(\{i \text{ has a value below the price for all items of } T_{i,v} \setminus \{j\}\} \mid C_1). \end{aligned}$$

Now the crucial observation is that the two events C_1 and "bidder i has a value below the price for all items in $T_{i,v} \setminus \{j\}$" are independent of each other: Whether $j \in A$ when bidder i is considered depends on the valuations of the earlier bidders and not on bidder i's valuations. Whether bidder i has a value above the price for no item in $T_{i,v} \setminus \{j\}$ only depends on valuations of bidder i and *not* on the valuations of the other bidders. Thus,

$$\begin{aligned} &P(\{i \text{ has a value below the price for all items of } T_{i,v} \setminus \{j\} \mid C_1\}) \\ &= P(\{i \text{ has a value below the price for all items of } T_{i,v} \setminus \{j\}\}) \geq \\ &\prod_{j' \in T_{i,v} \setminus \{j\}} (1 - q_{i,j'}) \geq 1 - \sum_{j \in T_{i,v} \setminus \{j\}} q_{i,j'} \geq \tfrac{1}{2} \end{aligned}$$

The last inequality holds because $\sum_{j' \in T_{i,v} \setminus \{j'\}} q_{i,j'} \leq \sum_{j' \in T_{i,v}} q_{i,j'} \leq 1/2$.

Consequently the total expected revenue of the algorithm is at least $\sum_i \sum_j P(C_1) \cdot P(C_2) \cdot R_{ij} \geq \sum_i \sum_j 3/4 \cdot 1/2 \cdot R_{ij} = 3/8 \cdot \sum_{i,j} R_{ij}$. The objective of LP3 is at most $4 \sum_{i,j} R_{ij}$ In other words the algorithm gives a 3/32-approximation to the objective of LP3.

□

Proof of ***Theorem 7 for a global matroid***. The difference in the proof between individual and global matroid constraints is that in the latter setting (a) the partition of the graph is done globally for all vertices and *not* individually for each vertex and (b) the budget constraint cannot be reasoned about for each bidder individually and, thus, we need to use a Markov bound argument that leads to a worse, but still constant competitive ratio. For completeness we give the full proof below.

We first solve LP3 to get the x_{ij} values and then decompose $x_{ij}/4$ according to Lemma 6 to get p_{ij}, r^*_{ij} and q_{ij}. By Lemma 6 and the first constraint of LP1 it follows that $\sum_i \sum_j q_{ij} = \sum_i \sum_{j \in S} \sum_r (x_{ij}(r)/4) f_{ij}(r) \leq \text{rank}(S)/4$ for each i and each subset S of J.

Let $G = (V, J)$ be the graph defining the feasibility constraint. The idea is to partition the matroid into sets T_v so that only a single item from each T_v is allocated and still obtain a good approximation to the expected optimal revenue. We can achieve this if we can find a partition of the items (which are the edges of G), into sets T_v such that $\cup_{v\in V} T_v = J$ and $\sum_i \sum_{j\in T_v} q_{ij} \leq 1/2$.

We can find such a partition as follows: For each $v \in V$ let $\delta(v)$ be the set of edges incident to v and define $q_v := \sum_i \sum_{j\in\delta(v)} q_{ij}$. Now $\sum_v q_v = 2\sum_i \sum_j q_{ij} \leq 2\cdot \text{rank}(V)/4 \leq (|V|-1)/2$. Thus there exists a v for which $q_v \leq 1/2$. We run the following algorithm in order to partition the graphical matroid.

1. $U = V$, $A = J$. While there exists a $v \in U$
   (a) For every node v set $q_v = \sum_i \sum_{j\in\delta(v)\cap A} q_{ij}$
   (b) Pick a node $v \in U$ with $q_v \leq 1/2$,
   (c) Set $T_v := \delta(v) \cap A$,
   (d) Set $U = U \setminus \{v\}$ and $A = A \setminus T_v$.

The sets T_v are chosen in such a way that any set that contains at most one element from T_v for each v, is guaranteed to be an independent set in $\mathcal{M}$, i.e., the matroid constraints are fulfilled. Now that we have the desired partition we can present our algorithm. Let $\tilde{r}_{ij}$ be the random variable that is equal to r^*_{ij} with probability p_{ij} and equal to $r^*_{ij}+1$ with probability $1-p_{ij}$.

1. $A = J$.
2. Order the bidders arbitrarily.
3. For $i = 1, 2, \ldots, n$
   (a) For every item $j \in A$ post for it a price of $\tilde{r}_{ij}$.
   (b) Ask bidder i to pick the items he prefers (without violating his budget constraints) with the restriction that he can only take a single item from each set $T_v \cap A$. Denote by S_i the set of elements picked by bidder i and set $A = A \setminus \cup_{v\in S_i} T_v$.

Let $OPT = \sum_{i,j} R_{ij}$. We will show that the revenue achieved by our mechanism is a constant factor of OPT. More specifically for each vertex v we will show that our mechanism expects to obtain a constant fraction of the revenue that the optimal solution extracts from T_v. We first define the following events.

1. Let C_1 denote the event $\{j \in A$ when bidder i is considered$\}$,

2. let C_2 denote the event $\{$bidder i picks no item different from j from $T_v \cap A\}$,

3. let C_3 denote the event $\{\sum_j P_{ij} < 3B_i/4\}$, which guarantees that player i has enough budget to take one more item, and

4. let C_4 denote the event $\{i$ has a value below the price for all items of $T_v \setminus \{j\}\}$.

Note that the events C_1, C_2 and C_3 are not independent of each other. If $A \cap T_v \neq \emptyset$ when bidder i is considered, then bidder i will pick an element j of $A \cap T_v$, provided $v_{ij} \geq \tilde{r}_{ij}$. Now Fix a set T_v and an item $j \in T_v$. If $j \in T_v \cap A$ and $T_v \cap A$ does not contain any other item that he prefers to item j, then we are sure that the expected revenue extracted is R_{ij}. Thus the expected revenue our algorithm extracts from each T_v is at least $\sum_{j\in T_v} P(C_1 \cap C_2 \cap C_3)\cdot R_{ij} = \sum_{j\in T_v} P(C_1)\cdot P(C_2 \cap C_3 \mid C_1)\cdot R_{ij}$ and the total expected revenue of the algorithm is at least $\sum_i \sum_v \sum_{j\in T_v} P(C_1)\cdot P(C_2\cap C_3 \mid C_1)\cdot R_{ij}$ $= \sum_i \sum_j P(C_1)\cdot P(C_2 \cap C_3 \mid C_1)\cdot R_{ij}$.

We show below that $P(C_1) \geq 3/4$ and $P(C_2 \cap C_3 \mid C_1) \geq 1/6$. By Lemma 6 and the constraints of LP3 we have that for each j, $E[\sum_{i'<i} q_{i'j}] \leq E[\sum_i \sum_r f_{ij}(r) x_{ij}(r)/4] \leq 1/4$ and consequently by the Markov inequality $P[\sum_i q_{ij} < 1] \geq 3/4$.

It holds that

$P(C_2 \cap C_3 \mid C_1) \geq P(\{i$ has a value below the price for all items of $(T_v \cap A) \setminus \{j\}\} \cap C_3 \mid C_1) \geq$ $P(\{i$ has a value below the price for all items of $T_v \setminus \{j\}\} \cap C_3 \mid C_1) = P(C_4 \cap C_3 \mid C_1)$.

Now the crucial observation is that the two events C_1 and $C_4 \cap C_3$ are independent of each other: Event C_1 depends on the valuations of the earlier bidders and not on bidder i's valuations. Events C_4 and C_3 only depend on valuations of bidder i and *not* on the valuations of the other bidders. Thus, $P(C_4 \cap C_3 \mid C_1) = P(C_4 \cap C_3)$. As before $P(C_4) \geq 1/2$ since $\sum_{j' \in T_v \setminus \{j\}} q_{i,j'} \leq \sum_{j \in T_v} q_{i,j} \leq 1/2$. By the second constraint of LP3 and Lemma 6 we get that for all i, $B_i/4 \geq \sum_j R_{ij}/4$. From the Markov inequality we get $P(C_3) = P(\sum_j R_{ij}/4 \geq 3B_i/4) \leq 1/3$. Thus, by union bounds $P(C_4 \cap C_3) = 1 - P(C_4 \cup C_3) \geq 1 - P(C_4) - P(C_3) \geq 1 - 1/2 - 1/3 = 1/6$

Consequently the total expected revenue of the algorithm is at least

$$\sum_i \sum_j P(C_4 \cap C_3) \cdot P(C_1) \cdot R_{ij} \geq \sum_i \sum_j 1/6 \cdot 3/4 \cdot R_{ij} = 1/8 \cdot \sum_{i,j} R_{ij}.$$

The objective of LP3 is at most $4 \sum_{i,j} R_{ij}$ In other words the algorithm gives a 32-approximation to the objective of LP3. □

If there are no budget constraints and a global matroid constraint then we can obtain a better approximation ratio with use of prophet inequalities, a very interesting technique introduced in [8, 20].

Theorem 8. *If for all i and j, $v_{ij} \in [1, L]$ follow independent distributions f_{ij} and satisfy the monotone hazard rate condition and there are no budget constraints, then a sequential posted price mechanism achieves a 3-approximation to LP3, and a $24e^2$-approximation of the revenue of the optimum BIC mechanism under a global matroid constraint.*

Proof. We first solve LP3 to get the x_{ij} values and then decompose x_{ij} according to Lemma 6 to get p_{ij}, r^*_{ij} and q_{ij} values. For each bidder i and each subset S of j it follows that $\sum_i \sum_j q_{ij} = \sum_i \sum_{j \in S} \sum_r x_{ij}(r) f_{ij}(r) \leq \text{rank}(S)$ where the equality is by Lemma 6 and the inequality from the LP constraints.

round x_{ij}.)

Partition the graph in the same way as in the previous proof into sets T_v. Recall that the sets T_v are chosen in such a way that if a set S contains no more than one element from each T_v then S is an independent set, i.e., the matroid constraints are fulfilled and that $\sum_i \sum_{j \in T_v} q_{ij} \leq 2$. For every bidder i and item j set $\tilde{r}_{ij} = r^*_{ij}$ with probability p_{ij} and set $\tilde{r}_{ij} = r^*_{ij} + 1$ with probability $1 - p_{ij}$.

Let $t_v := \sum_i \sum_{j \in T_v} \Big(p_{ij} P(\mathcal{V}_{ij} \geq r^*_{ij})(r^*_{ij} - t_v)^+ + (1 - p_{ij}) P(\mathcal{V}_{ij} \geq r^*_{ij} + 1)(r^*_{ij} + 1 - t_v)^+ \Big)$ where $(a)^+ := a$ if $a > 0$ and 0 else. Use the following mechanism that discard the rest of the items in T_v as soon as one element from T_v is allocated.

1. $A = J$.
2. Order the bidders arbitrarily.
3. For $i = 1, 2, \ldots, n$
   (a) For every item $j \in A$ offer it to player i at a price $\tilde{r}_{ij}$ but only if $\tilde{r}_{ij} \geq t_v$.

(b) Ask bidder i to pick the items he prefers, but with the restriction that he can only take a single item from each set $T_v \cap A$. Denote by S_i the set of elements picked by bidder i and and remove from A *the whole set* T_v, for all v such that the player picked an element from T_v, i.e. if $T_v \cap S_i \neq \emptyset$.

We will first give an upper bound of $3\sum_v t_v$ on the optimal expected revenue in terms of the threshold t_v we defined. Note that $a + (b-a)^+ = \max\{a, b\}$.

$$
\begin{aligned}
OPT &\leq \sum_{i,j} p_{ij} P(\mathcal{V}_{ij} \geq r^*_{ij}) r^*_{ij} + \sum_{i,j} (1-p_{ij}) P(\mathcal{V}_{ij} \geq r^*_{ij}+1)(r^*_{ij}+1) \\
&\leq \sum_v \sum_i \sum_{j \in T_v} p_{ij} P(\mathcal{V}_{ij} \geq r^*_{ij}) \max\{t_v, r^*_{ij}\} + \\
&\quad \sum_v \sum_i \sum_{j \in T_v} (1-p_{ij}) P(\mathcal{V}_{ij} \geq r^*_{ij}+1) \max\{t_v, r^*_{ij}+1\} \\
&= \sum_v \sum_i \sum_{j \in T_v} p_{ij} P(\mathcal{V}_{ij} \geq r^*_{ij})(t_v + (r^*_{ij} - t_v)^+) + \\
&\quad \sum_v \sum_i \sum_{j \in T_v} (1-p_{ij}) P(\mathcal{V}_{ij} \geq r^*_{ij}+1)(t_v + ((r^*_{ij}+1) - t_v)^+) \\
&= \sum_v \sum_i \sum_{j \in T_v} \left[p_{ij} P(\mathcal{V}_{ij} \geq r^*_{ij}) + (1-p_{ij}) P(\mathcal{V}_{ij} \geq r^*_{ij}+1)\right] t_v + \sum_v t_v \\
&= \sum_v t_v \sum_i \sum_{j \in T_v} q_{ij} + \sum_v t_v \leq 3 \sum_v t_v \qquad \left(\text{as } \sum_i \sum_{j \in T_v} q_{ij} \leq 2\right)
\end{aligned}
$$

We now show that the average revenue of our algorithm is at least $\sum_v t_v$. We concentrate on a single T_v and show that our algorithm expects to receive a revenue of t_v. The first bidder whose valuation v_{ij} satisfies $v_{ij} \geq \tilde{r}_{ij} \geq t_v$ for some item j is offered the item. The revenue that our algorithm gets is then $\tilde{r}_{ij}$.

Let H_{ij} denote the event that "no item of T_v has been picked by a bidder $i' < i$ and i has not picked an item different from j of T_v" and let H_∞ denote the event that "no item of T_v has been picked by any bidder". Let also R^v_{ij} denote the revenue of our algorithm in T_v if bidder i picks item j.

$$
\begin{aligned}
E[\text{revenue of our mechanism in } T_v] &= (1 - P[H_\infty]) \cdot t_v + \sum_{i, j \in T_v} E[(R^v_{ij} - t_v)^+ \mid H_{ij}] \cdot P[H_{ij}] \\
&\geq (1 - P[H_\infty]) \cdot t_v + P[H_\infty] \cdot \sum_{i, j \in T_v} E[(R^v_{ij} - t_v)^+ \mid H_{ij}] \\
&\geq (1 - P[H_\infty]) \cdot t_v + P[H_\infty] \cdot E[\sum_{i, j \in T_v} ((R^v_{ij} - t_v)^+] \\
&= (1 - P[H_\infty]) \cdot t_v \\
&\quad + P[H_\infty] \cdot \sum_i \sum_{j \in T_v} \Big(p_{ij} P(\mathcal{V}_{ij} \geq r^*_{ij})(r^*_{ij} - t_v)^+ + (1-p_{ij}) P(\mathcal{V}_{ij} \geq r^*_{ij}+1)(r^*_{ij}+1-t_v)^+\Big) \\
&= (1 - P[H_\infty]) \cdot t_v + P[H_\infty] \cdot t_v = t_v
\end{aligned}
$$

For the first inequality note that at any step of the algorithm we have $P[H_{i,j}] \geq P[H_\infty]$. The second inequality is because the valuations of different bidders and different items are independent. Consequently the algorithm gives a 3-approximation to the objective of LP3. □

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Appendix A. Appendix

Proof of ***Theorem 3 for both Individual and Global matroids***. Since the proof is practically the same for Individual and global matroids we simply note in parentheses the points where the proof for global matroids differs.

For each bidder i in the case of Individual matroids it follows that

$$\sum_j q_{ij} = \sum_j \sum_r (x_{ij}(r)/(2k)) f_{ij}(r) \leq \text{rank}(J)/2k = 1/2$$

where the equality is by Lemma 6 and the inequality from the first constraint of LP3. (For global matroids it holds that $\sum_i \sum_j q_{ij} = \sum_i \sum_j \sum_r (x_{ij}(r)/(2k)) f_{ij}(r) \leq \text{rank}(J)/2k = 1/2$.)

Note that the optimal solution to LP3 achieves a value of $2k \sum_{i,j} R_{i,j}$ and by Lemma 2 and 5 the revenue of the optimal BIC mechanism is $O(k \sum_{ij} R_{i,j})$. We will show that the revenue achieved by our mechanism is a constant factor of R_{ij} for each bidder item pair (i, j). By Lemma 6, $\tilde{r}_{ij} \leq B_i/4$ so that bidder i will not be budget constrained. Also the matroid constraints pose no challenge: If bidder i cannot take item j because of the matroid constraints, then no independent set can contain j and thus the optimal mechanism cannot get any revenue from j for bidder i either. Thus it suffices to consider items j that belong to at least one independent set of $\mathcal{M}_i$ (resp. $\mathcal{M}$ for global matroids).

Let C_1 denote the event $\{j \in A$ when bidder i is considered$\}$ and C_2 denote the event $\{$bidder i picks no item from $A \setminus \{j\}\}$. The expected revenue our algorithm extracts for bidder i and item j is at least $P(C_1 \cap C_2) \cdot R_{ij} = P(C_1) \cdot P(C_2 \mid C_1) \cdot R_{ij}$ while, due to the definition of $\tilde{x}_{ij}(r)$ the optimal mechanism extracts a revenue of at most $2k \cdot R_{ij}$. Thus it suffices to show that $P(C_1) \geq 1/2$ and $P(C_2 \mid C_1) \geq 1/2$ to show that our mechanism has an expected revenue of $R_{ij}/4$.

The probability $P(\{j \in A$ when bidder i is considered$\})$, i.e., the probability that j has not been allocated to any of the previous bidders, is 1/3, because from Lemma 6 and the first constraint of LP3 we have that for each j, $E[\sum_i q_{ij}] = E[\sum_i \sum_r f_{ij}(r) x_{ij}(r)] \leq 1/2$ and consequently by the Markov inequality $P[\sum_i q_{ij} \geq 1] \leq 1/2$ and, hence, $P[\sum_{i'<i} q_{ij} < 1] \geq P[\sum_i q_{ij} < 1] \geq 1/2$.

Additionally we have that $P(C_2 \mid C_1) = P(\{i$ picks no other item from $A \mid C_1\}) \geq P(\{i$ has a value below the price for all items of $A \setminus \{j\} \mid C_1\} \geq P(\{i$ has a value below the price for all items of $J \setminus \{j\} \mid C_1\}$. Now the crucial observation is that the two events C_1 and "bidder i has a value below the price for all items in $J \setminus \{j\}$" are independent of each other: Whether $j \in A$ when bidder i is considered depends on the valuations of the earlier bidders and not on bidder i's valuations. Whether bidder i has a value above the price for no item in $J \setminus \{j\}$ only depends on valuations of bidder i and *not* on the valuations of the other bidders. Thus, $P(\{i$ has a value below the price for all items of $J \setminus \{j\} \mid C_1\} = P(\{i$ has a value below the price for all items of $J \setminus \{j\}\}) \geq \prod_{j' \in J \setminus \{j\}} (1 - q_{i,j'}) \geq 1 - \sum_{j \in J \setminus \{j\}} q_{i,j'} \geq \frac{1}{2}$ since $\sum_{j \in J \setminus \{j\}} q_{i,j} \leq \sum_{j \in j} q_{i,j} \leq 1/2$.

In the case of global matroid constraints the formulas change slightly as follows:

$$P(\{i \text{ has a value below the price for all items of } J \setminus \{j\} \mid C_1\}) =$$
$$P(\{i \text{ has a value below the price for all items of} J \setminus \{j\}\} \geq$$
$$\prod_{j' \in J \setminus \{j\}} (1 - q_{i,j'}) \geq 1 - \sum_{j \in J \setminus \{j\}} q_{i,j'} \geq \frac{1}{2} \qquad (\text{since} \sum_{j \in J \setminus \{j\}} q_{i,j} \leq \sum_i \sum_j q_{i,j} \leq 1/2).$$

□

Lemma 7. *Let v be an integer valued random variable $v \in [1, L]$, let $f(t) := P[v = t]$ and suppose that its virtual valuation $\phi(t)$ satisfies the MHR condition. Then $P[\phi(t) \geq \frac{t}{2}] \geq \frac{1}{e^2}$.*

Proof. Let $h(t) = \frac{f_{ij}(t)}{1 - F_{ij}(t)}$ denote the hazard rate of f. We assumed that f satisfies MHR, which means that h is non-decreasing and from this it follows that $\phi(t) - t/2$ is also a non-decreasing function of t. The function $\phi(t) - t/2$ is defined for $t \in [1, L]$ and it holds that $\phi(L) - L/2 = L/2 > 0$ (since $F(L) = 1$). Consequently either $\phi(t) \geq t/2$ for all $t \in [1, L]$ or there exists some integer k with $1 \leq k \leq L$, such that $t > k$ if and only if $\phi(t) \geq t/2$. In the first case we are done since $P[\phi(t) \geq \frac{t}{2}] = 1$. In the second case it holds that $P[\phi(t) \geq \frac{t}{2}] \geq P[t > k]$.

From our initial assumptions $f(t)$ is a discrete distribution defined in $[1, L]$. Let now $\hat{f}$ be a continuous distribution whose density function is defined as follows: For any integer $t \in [1, L]$ we have that the value of $\hat{f}(t)$ in the interval $[t, t+1)$ is constant and equal to $f(t)$. The $\hat{f}$ we construct is defined in $[1, L+1)$. Let $\hat{F}(t) = \int_{q=1}^{t} \hat{f}(q) dq$ and $\hat{h}(t) = \frac{\hat{f}(t)}{1 - \hat{F}(t)}$.

Since $\hat{F}(1) = \int_1^1 \hat{f}(q)dq = 0$ and

$$\int_{t=1}^{k} \hat{h}(t)dt = \int_{t=1}^{k} \frac{\hat{f}(t)}{1-\hat{F}(t)}dt = -\int_{t=1}^{k} \frac{(1-\hat{F}(t))'}{1-\hat{F}(t)}dt = -\sum_{q=1}^{k-1}\int_{t=q}^{q+1} \frac{(1-\hat{F}(t))'}{1-\hat{F}(t)}dt$$
$$= -\sum_{q=1}^{k-1}[\ln(1-\hat{F}(t))]_q^{q+1} = -\ln(1-\hat{F}(k)) + \ln(1-\hat{F}(1)) = -\ln(1-\hat{F}(k)),$$

we have $1-\hat{F}(k) = e^{-\int_{t=1}^{k}\hat{h}(t)dt}$.

Moreover for all integers k we have $\hat{F}(k+1) = \int_1^{k+1}\hat{f}(t)dt = \sum_{q=1}^{k} f(q)\int_q^{q+1} dt = \sum_{q=1}^{k} f(q) = F(k)$. Note that as $\phi(k) = k - \frac{1}{h(k)}$ and $\phi(k) \le \frac{k}{2}$ we get that $h(k) \le \frac{2}{k}$. Since h is non-decreasing it follows that for all $t \le k$, $h(t) \le h(k) \le \frac{2}{k}$. If q is an integer and $t \in [q, q+1)$, we have $\hat{F}(t) \le F(q)$ and $\hat{f}(t) = f(q)$. It follows that $\hat{h}(t) \le h(q)$, which gives $\hat{h}(t) \le \frac{2}{k}$ for all $t < k+1$. Using the previous two observations we get:

$$P[\phi(t) \ge \frac{t}{2}] = P[t > k] = 1 - F(k) = 1 - \hat{F}(k+1) = e^{-\int_{t=1}^{k+1}\hat{h}(t)dt} \ge e^{-\frac{2}{k}\int_{t=1}^{k+1} dt} = e^{-2}.$$

□

of Lemma 5.

Claim 9. *If v is a random variable satisfying the MHR condition then $\min\{v, a\}$ satisfies the MHR condition for any integer $a \ge 1$. Consequently the distribution of the random variable $\mathcal{V}_{ij} = \min\{v_{ij}, B_i/4\}$ satisfies the MHR condition, since the distribution of v_{ij} satisfies MHR.*

Proof. We start with a non-decreasing function h and when the function exceeds the value a you make it constant. The new function is obviously non-decreasing as well. □

Let $v_{ij}^* := \text{argmin}_r\{\phi_{ij}(r) \ge \frac{r}{2}\}$. Then by Lemma 7 and the previous Claim we have $P[\mathcal{V}_{ij} \ge v_{ij}^*] \ge \frac{1}{e^2}$.

Let $OBJ = \sum_r r f_{ij}(r)x_{ij}(r)$, let $L = \sum_{r<v_{ij}^*} r f_{ij}(r)x_{ij}(r)$ be the terms of the sum that correspond to $r < v_{ij}^*$ (we will argue that ignoring these we do not loose more than a factor of e^2 in the objective function) and let $H = \sum_{r\ge v_{ij}^*} r f_{ij}(r)x_{ij}(r)$ be the remaining terms of the sum. We want to show that $H \ge \frac{OBJ}{e^2}$. Indeed since x_{ij} is a non-decreasing function of r,

$$H = \sum_{r\ge v_{ij}^*} r f_{ij}(r)x_{ij}(r) \ge v_{ij}^* x_{ij}(v_{ij}^*)\sum_{r\ge v_{ij}^*} f_{ij}(r) \ge v_{ij}^* x_{ij}(v_{ij}^*)\cdot P[\mathcal{V}_{ij} \ge v_{ij}^*] \ge v_{ij}^* x_{ij}(v_{ij}^*)\cdot\frac{1}{e^2},$$

where the latter inequality is by Lemma 7. Similarly we have

$$L = \sum_{r<v_{ij}^*} r f_{ij}(r)x_{ij}(r) < v_{ij}^* x_{ij}(v_{ij}^*)\sum_{r<v_{ij}^*} f_{ij}(r) = v_{ij}^* x_{ij}(v_{ij}^*)\cdot P[\mathcal{V}_{ij} < v_{ij}^*] \le v_{ij}^* x_{ij}(v_{ij}^*)(1-\frac{1}{e^2}).$$

Putting both these inequalities together we get $\frac{OBJ}{H} = \frac{L+H}{H} = \frac{L}{H} + 1 \le e^2$.

Now replacing r by $\phi_{ij}(r) \in [r/2, r]$ for $r \ge v_{ij}^*$ preserves the constraints (since $\phi_{ij}(r) \le r$) and looses another factor of 2 in the objective function since $\phi(r) \ge r/2$. □

Proof of Lemma 6. For each pair (i,j) we can decrease $x_{ij}(r)$ for small r and increase it for big r while preserving $\sum_r x_{ij}(r)\phi_{ij}(r)f_{ij}(r)$. We have $x_{ij}=0$ for $r<r^*_{ij}$ and 1 for $r>r^*_{ij}$ for some r^*_{ij} with $1\le r^*_{ij}<r^*_{ij}+1\le B_i/4$, where to derive the latter inequality we use the fact that $B_i>8$ and $\mathcal{V}_{ij}:=\min\{v_{ij},B_i/4\}$. so that the decomposition in the statement of the Lemma holds for $p_{ij}=x_{ij}(r^*_{ij})$. Since we assumed that the distribution satisfies MHR, ϕ_{ij} is a non-decreasing function of r and so $\sum_r x_{ij}(r)\phi_{ij}(r)f_{ij}(r)$ will decrease after modifying $x_{ij}(r)$, so all constraints will be preserved.

(a) From Myerson's characterization [17] we have

$$\sum_r f_{ij}(r)\phi_{ij}(r)=\sum_r f_{ij}(r)p_{ij}(r)=\sum_r\Big(f_{ij}(r)\big(rx_{ij}(r)-\sum_{s=1}^{r-1}x_{ij}(s)\big)\Big)$$

Using that by definition $y_{ij}(r)=0$ for $r<r^*_{ij}$ and then that again by definition $y_{ij}(r)=1$ for $r\ge r^*_{ij}$ we have

$$\begin{aligned}\sum_r\big(f_{ij}(r)\big(ry_{ij}(r)-\sum_{s=1}^{r-1}y_{ij}(s)\big)\big) &= \sum_{r\ge r^*_{ij}} rf_{ij}(r)y_{ij}(r)-\sum_{r\ge r^*_{ij}} f_{ij}(r)\sum_{s=r^*_{ij}}^{r-1}y_{ij}(s)\\ &= \sum_{r\ge r^*_{ij}} rf_{ij}(r)-\sum_{r\ge r^*_{ij}} f_{ij}(r)(r-r^*_{ij})\\ &= \sum_{r\ge r^*_{ij}} f_{ij}(r)r^*_{ij}\\ &= r^*_{ij}\cdot P[\mathcal{V}_{ij}\ge r^*_{ij}].\end{aligned}$$

Similarly $\sum_r\big(f_{ij}(r)\big(rz_{ij}(r)-\sum_{s=1}^{r-1}z_{ij}(r)\big)\big)=(r^*_{ij}+1)P[\mathcal{V}_{ij}\ge r^*_{ij}+1]$ and combining the two equations we get the desired result.

(b) Note that as $x_{ij}(r)=p_{ij}y_{ij}(r)+(1-p_{ij})z_{ij}(r)$ and from the definition of y_{ij},z_{ij}, it follows that $x_{ij}(r^*_{ij})=1$ for all $r\ge r^*_{ij}+1$ and $x_{ij}(r)=0$ for all $r<r^*_{ij}$. We have:

$$\begin{aligned}q_{ij} &= p_{ij}P(\mathcal{V}_{ij}\ge r^*_{ij})+(1-p_{ij})P(\mathcal{V}_{ij}\ge r^*_{ij}+1)\\ &= p_{ij}\sum_{r\ge r^*_{ij}} f_{ij}(r)+(1-p_{ij})\sum_{r\ge r^*_{ij}+1} f_{ij}(r)\\ &= p_{ij}f_{ij}(r^*_{ij})+\sum_{r\ge r^*_{ij}+1} f_{ij}(r)\\ &= x_{ij}(r^*_{ij})f_{ij}(r^*_{ij})+\sum_{r\ge r^*_{ij}+1} x_{ij}(r)f_{ij}(r)+\sum_{r<r^*_{ij}} x_{ij}(r)f_{ij}(r)\\ &= \sum_r x_{ij}(r)f_{ij}(r).\end{aligned}$$

□