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A complete characterization of group-strategyproof mechanisms of cost-sharing

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Abstract We study the problem of designing group-strategyproof cost-sharing mechanisms. The players report their bids for getting serviced and the mechanism decides a set of players that are going to be serviced and how much each one of them is going to pay. We determine three conditions: *Fence Monotonicity*, *Stability* of the allocation and *Validity* of the tie-breaking rule that are necessary and sufficient for group-strategyproofness, regardless of the cost function. Consequently, Fence Monotonicity characterizes group-strategyproof cost-sharing schemes closing an important open problem. Finally, we use our results to prove that there exist families of cost functions, where any group-strategyproof mechanism has arbitrarily poor budget balance.

1 Introduction

Algorithmic Mechanism Design [1] is a field of Game Theory, that tries to construct algorithms for allocating resources that give to the players incentives to report their true interest in receiving a good, a service, or in participating in a given collective activity. The pivotal constraint when designing a mechanism for any problem is that it is truthful. Truthfulness also known as strategyproofness requires that no player can strictly improve her utility by lying. In many

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settings this single requirement for an algorithm to be truthful restricts the repertoire of possible algorithms dramatically [2–5].

In settings, where truthfulness does not impose such severe limitations, like for example in cost-sharing problems, it is desirable to construct mechanisms that are also resistant to manipulation by groups of players. A group of players forms a *successful coalition* when the utility of *each* player in the group does not decrease and the utility of *at least one* player *strictly* increases if some of them announce values other than their true value. Group-strategyproofness naturally generalizes truthfulness by requiring that no group of players can form a successful coalition by lying, when the values of the other players are fixed.

In this paper we study the following problem: A set of n customers/players are interested in receiving a service. Players report their valuation of the service and the mechanism decides the players that are going to receive the service and the price that each one of them will pay. We want to characterize all possible mechanisms that satisfy group-strategyproofness via identifying necessary and sufficient conditions for the payment functions that give rise to these mechanisms.

We provide a complete characterization of group-strategyproof mechanisms closing an important open question posed in [6], [7, Chapter 15]. Immorlica, Mahdian and Mirrokni [6] identified the property of semi-cross-monotonicity, a local property that should be satisfied by the payment part (cost-sharing scheme) of every group-strategyproof mechanism. We introduce a generalization of semi-cross-monotonicity, Fence Monotonicity, which still refers only to the payment part of the mechanism, and is not only *necessary* but also *sufficient* for group-strategyproofness. Given any payment rule that satisfies Fence Monotonicity, we show that every allocation that satisfies *Stability* and *Validity* of the tie-breaking rule, yields a group-strategyproof mechanism. Our characterization of group-strategyproofness works for any cost function as we do not make any assumption about budget balance. Also, there are no additional constraints like fixed tie-breaking rules [6, 8].

Our results provide a new general framework for designing group-strategyproof mechanisms. Thus, it opens new perspectives to the study of the very important problem of cost-sharing, the study of which was initiated by Moulin [9], where we additionally have a cost function C such that for each subset of players S the cost for providing service to all the players in S is $C(S)$. However, the strength of our results is that they apply to any cost function, since throughout our proof we do not make any assumptions at all about this cost function. We believe that our work can be the starting point for constructing new interesting classes of mechanisms for specific cost-sharing problems.

As our characterization is the first characterization of group-strategyproofness, it is interesting to draw a parallel between our result and the many known characterizations of strategyproofness [10, 11, 2–5] and our characterization of group-strategyproofness. Fence Monotonicity is a condition rather similar to Cycle Monotonicity [10], in the sense that both are conditions that should be satisfied by all possible restrictions of the output space of the mechanism. A great virtue for a characterization of a mechanism that uses money is to be able to separate the payment from the allocation part. Cycle Monotonicity is a necessary property for the allocation. If it is satisfied, we know a way to define truthful payments [10]. Fence Monotonicity refers only to the payments of a mechanism and if it is satisfied, Stability and Validity of the tie-breaking rule can be used to determine a group-strategyproof allocation.

We believe that Fence Monotonicity can also be a starting point in the quest for alternative characterizations, as Cycle monotonicity has done so far [11, 2–5]. Unfortunately, even though Fence Monotonicity is succinct in its description, it is much more complicated than Cycle Monotonicity. Nonetheless, group-strategyproofness is a notion much more complicated than strategyproofness and we believe that Fence Monotonicity is not only important and unavoidable but also useful. We demonstrate the latter by proving the first, even though simple, lower bound on the budget balance of any group-strategyproof mechanism.

1.1 Our results and related work.

The design of group-strategyproof mechanisms for cost-sharing was first discussed by Moulin and Shenker [9, 12]. Moulin [9] defined a condition on the payments called cross-monotonicity, which states that the payment of a serviced player should not increase as the set of serviced players grows. Any mechanism with payments satisfying cross-monotonicity can be easily turned to a simple mechanism named after Moulin. A Moulin mechanism first checks if it can provide service to all players, so that each one has non-negative utility and if not, it gradually diminishes the set of players that are candidates to be serviced, by throwing away at each step a player that cannot pay to get serviced (and who because of cross-monotonicity still cannot pay if the set of candidates becomes smaller). In fact, if the cost function is sub-modular, then the only possible 1-budget balanced group-strategyproof mechanisms are Moulin mechanisms [12]. The great majority of cost-sharing mechanisms proposed are Moulin mechanisms.

Nevertheless, recent results showed that for several important cost-sharing games Moulin mechanisms can only achieve a very bad budget balance factor [6, 13, 14]. Some alternative, very interesting, and much more complicated in their description mechanisms that are group-strategyproof but not Moulin have been proposed [6, 15], however, these do not exhaust the class of group-strategyproof mechanisms. In this work we introduce *Fencing Mechanisms*, a new general framework for designing group-strategyproof mechanisms, that generalizes Moulin mechanisms [9].

Recently, Mehta et. al. [16] proposed the notion of weak group-strategyproofness, that relaxes group-strategyproofness. It regards a formation of a coalition, as successful, when *each* player who participates in the coalition *strictly* increases her personal utility. They also introduce acyclic mechanisms, a general framework for designing weakly group-strategyproof mechanisms, however, the question of determining all possible weakly group-strategyproof mechanisms is an important question that remains open. Another alternative notion of group-strategyproofness was proposed by Bleischwitz et. al. in [17].

The problem we solve here was a major open problem posed in [6],[7, Chapter 15]. The attempt to find such a characterization has lead to several interesting results [15,8]. In contrast to previous characterization attempts that characterized mechanisms satisfying some additional boundary constraints [6, 8], our characterization is complete and succinct. The only previously known complete characterization was for the case of two players [15,8]. The great challenge now is how our characterization can be applied for constructing new efficient mechanisms for specific cost-sharing problems. We believe that it can significantly enrich the repertoire of mechanisms with good budget balance guarantees for specific problems.

In the notion of group-strategyproofness it is important to understand that ties play a very important role. This is in contrast to strategyproofness, where ties can be in most cases broken arbitrarily (see for example [18]). An intuitive way to understand this is that a mechanism designer of a group-strategyproof mechanism expects a player to tell a lie in order to help the other players increase their utility, even when she would not gain any profit for herself. This player is at a tie but decides strategically if she should lie or not. Consequently, a characterization that assumes a priori a tie-breaking rule, and hence, greatly restricts the repertoire of possible mechanisms [6,19] might be useful for specific problems and easier in its statement, but can never capture group-strategyproofness in its full generality.

Our proofs are involved and make a repeated use of induction. The main difficulty and value of our work though, was to identify necessary and sufficient conditions for group-strategyproof payments, that are also succinct to describe and add to our understanding of the notion of group-strategyproofness. In proving the necessity of Fence Monotonicity we first have to prove lemmas that also reveal interesting properties of the allocation part of the mechanism. A novel tool that we introduce is the *harm relation* that refines the notion of negative elements defined in [6]. Proving that Fencing Mechanisms, i.e., mechanisms with payments satisfying Fence Monotonicity and allocation satisfying Stability and Validity of the tie-breaking rule, are group-strategyproof turns out to be rather complicated.

2 Defining the model

The Mechanism

Suppose that $\mathcal{A} = \{1, 2, \ldots, n\}$ is a set of players interested in receiving a service. Each of the players has a private type v_i, which is her valuation for receiving the service.

A cost sharing mechanism (O, p) consists of a pair of functions, $O : \mathbb{R}^n \to 2^{\mathcal{A}}$ that associates with each bid vector b the set of serviced players and p :

$\mathbb{R}^n \to \mathbb{R}^n$ that associates with each bid vector b a vector $p(b) = (p_1(b), \ldots, p_n(b))$, where the i-th coordinate is the payment of player i. Assuming quasi-linear utilities, each player wants to maximize the quantity $v_i a_i - p_i(b)$ where $a_i = 1$ if $i \in O(b)$ and $a_i = 0$ if $i \notin O(b)$.

As it is common in the literature, we concentrate on mechanisms that satisfy the following very simple conditions [9,12,6]:

- *Voluntary Participation (VP)*: A player that is not serviced is not charged ($i \notin O(b) \Rightarrow p_i(b) = 0$) and a serviced player is never charged more than her bid ($i \in O(b) \Rightarrow p_i(b) \leq b_i$).
- *No Positive Transfer (NPT)*: The payment of each player i is non-negative ($p_i(b) \geq 0$ for all i).
- *Consumer Sovereignty (CS)*: For each player i there exists a value $b_i^* \in \mathbb{R}$ such that if she bids b_i^*, then it is guaranteed that player i will receive the service no matter what the other players bid.[1]

In accordance with [9,6] we will concentrate on mechanisms, where the players are additionally able to deny receiving service, no matter what the other players bid. To ensure this we just assume that players can report negative bids. Then VP and NPT imply that if a player announces a negative amount, she will not receive the service. Even though negative bids may not seem realistic, they can model the denial of revealing any information to the mechanism. To put things simply, if a player i bids -1, then she does not receive service and if she bids b_i^*, then she receives service, no matter what the other players bid.

We are interested in mechanisms that are *group-strategyproof (GSP)*. A mechanism is GSP if for every two valuation vectors v, v' and every coalition of players $S \subseteq \mathcal{A}$, satisfying $v_i = v_i'$ for all $i \notin S$, one of the following is true: *(a)* There is some $i \in S$, such that $v_i a_i' - p_i(v') < v_i a_i - p_i(v)$ or *(b)* for all $i \in S$, it holds that $v_i a_i' - p_i(v') = v_i a_i - p_i(v)$.

A *cost-sharing scheme* is a function $\xi : \mathcal{A} \times 2^{\mathcal{A}} \to \mathbb{R}^+ \cup \{0\}$ such that for every $S \subset \mathcal{A}$ and every $i \notin S$, we have $\xi(i, S) = 0$. Theorem 4.1 (c) in Immorlica et. al. [6] states that for any GSP mechanism (O, p) of our setting there is an underlying cost-sharing scheme ξ such that for every bid vector b and player i it holds, that $p_i(b) = \xi(i, O(b))$. In other words the payment of an agent depends only on the outcome and may not be different at two bid vectors that are mapped by the allocation of the mechanism to the same subset.

The cost function and budget balance.

The cost of providing service is given by a cost function $C : 2^{\mathcal{A}} \to \mathbb{R}^+ \cup \{0\}$, where $C(S)$ specifies the cost of providing service to all players in S.

A desirable property of cost-sharing schemes with respect to some cost function is budget balance. We say that a cost-sharing scheme ξ is *α-budget*

[1] From VP it holds that $b_i^* \geq \max_{b \in \mathbb{R}^n} p_i(b)$. It is easy to verify that strategyproofness implies that any value greater than b_i^* satisfies CS for player i. Thus, when we refer to this crucial value, we will without loss of generality assume that this inequality is strict.

balanced, where $0 \leq \alpha \leq 1$, if for every $S \subseteq \mathcal{A}$ it holds that $\alpha \cdot C(S) \leq \sum_{i \in S} \xi(i,S) \leq C(S)$. Correspondingly, we say that a mechanism is α-budget balanced if its cost sharing scheme is α-budget balanced. We chose to define the cost function last in order to stress that our results are completely independent of any budget-balance assumption, thus, they apply to any cost-sharing problem.

An important and well-studied property of cost-sharing schemes is cross-monotonicity, which is sufficient for group-strategyproofness [9]. A cost-sharing scheme is *cross-monotonic* if $\xi(i,S) \geq \xi(i,T)$ for every $S \subset T \subseteq \mathcal{A}$ and every player $i \in S$. This means that the payment of a player cannot increase as the number of players that receive service increases.

In the attempt to provide a characterization of GSP mechanisms Immorlica et. al. [6] provided a partial characterization and identified semi-cross-monotonicity, an important condition that should be satisfied by the cost-sharing scheme of any GSP mechanism. A cost sharing scheme ξ is *semi-cross-monotonic* if for every $S \subseteq \mathcal{A}$, and every player $i \in S$, either for all $j \in S \setminus \{i\}$, $\xi(j, S \setminus \{i\}) \leq \xi(j,S)$ or for all $j \in S \setminus \{i\}$, $\xi(j, S \setminus \{i\}) \geq \xi(j,S)$.

As we later show in Proposition 1 (i), semi-cross-monotonicity can be almost directly derived from Fence Monotonicity, the property we introduce in this work.

3 Our Characterization

3.1 Fence Monotonicity

Fence Monotonicity considers each time a restriction of the mechanism that can only output as the serviced set, subsets of U that contain all players in L. To be more formal, consider all possible pairs of subsets of the players L, U such that $L \subseteq U \subseteq \mathcal{A}$. Given a pair $L \subseteq U$, Fence Monotonicity considers only sets of players S with $L \subseteq S \subseteq U$. Each Fence Monotonicity condition should be satisfied for any possible such restriction of the players that are candidates for receiving service. We call the condition Fence Monotonicity because we "fence" the possible allocations of the mechanism by the sets L and U.

We denote by $\xi^*(i, L, U)$ the minimum payment of player i for getting serviced when the output of the mechanism is restricted by L and U, i.e., $\xi^*(i,L,U) := \min_{\{L \subseteq S \subseteq U, i \in S\}} \xi(i,S)$.

Definition 1 (Fence Monotonicity) We will say that a cost-sharing scheme satisfies Fence Monotonicity, if for every $L \subseteq U \subseteq \mathcal{A}$, it satisfies the following three conditions:

(a) There exists one set S with $L \subseteq S \subseteq U$, such that for all $i \in S$, we have $\xi(i,S) = \xi^*(i,L,U)$.

(b) For each player $i \in U \setminus L$ there exists one set S_i with $i \in S_i$ and $L \subseteq S_i \subseteq U$, such that for all $j \in S_i \setminus L$, we have $\xi(j,S_i) = \xi^*(j,L,U)$.
(Since $i \in S_i \setminus L$, it holds that $\xi(i,S_i) = \xi^*(i,L,U)$.)

(c) If for some $C \subset U$ there is a player $j \in C$ with $\xi(j, C) < \xi^*(j, L, U)$ (obviously $L \nsubseteq C$), then there exists one set $T \neq \emptyset$ with $T \subseteq L \setminus C$, such that for all $i \in T$, $\xi(i, C \cup T) = \xi^*(i, L, U)$.

An alternative way of expressing conditions (a) and (b) of Fence Monotonicity at some L, U is the following: We say that a set S is *optimal* for a player $i \in S$ if she is serviced with her minimum possible payment in this restriction, i.e., $\xi^*(i, L, U) = \xi(i, S)$. Condition (a) requires the existence of a set that is optimal for all players in it. Condition (b) says that each agent $i \in U \setminus L$ belongs to some set S_i that is optimal for the players in $S_i \setminus L$. Notice that for two different players i, j the sets S_i, S_j might be different.

Condition (c) is the most important and involved condition. It compares the minimum possible payment of a player in this restriction of the mechanism with her payment when the outcome can be any subset of U, i.e., it should not necessarily contain the players in L. Assume that player j is better off when the outcome is some set C, where $L \nsubseteq C$, i.e., loosely speaking some of the players in $L \setminus C$ "harm" player j by their presence in the outcome. Then condition (c) requires the existence of some non-empty set T with $T \subseteq L \setminus C$, such that each player in T is indifferent between her payment in $C \cup T$ and her minimum payment in the original restriction.

Observe that if ξ is cross-monotonic, then the set U is optimal for every player. Hence, conditions (a) and (b) are always satisfied. Moreover, condition (c) is trivially satisfied, as for every $C \subset U$, it holds that for all $i \in C$, $\xi(i, C) \geq \xi(i, U) = \xi^*(i, L, U)$. As a result, cross-monotonicity implies Fence Monotonicity.

As we prove in Section 6 semi-cross-monotonicity imposes a restriction equivalent to a special case of property (b) of Fence monotonicity. Therefore, every cost sharing scheme that satisfies Fence Monotonicity also satisfies semi-cross-monotonicity. The following theorem is the main contribution of this paper. The only if and if parts of the theorem are proven in Sections 4 and 5 respectively.

Theorem 1 *A cost sharing scheme gives rise to a GSP mechanism if and only if it satisfies Fence Monotonicity.*

3.2 Fencing Mechanisms

If one is given a cost sharing scheme that satisfies Fence Monotonicity, it is not straightforward constructing a GSP mechanism. Fence Monotonicity can yield a GSP mechanism if and only if it is coupled with an allocation rule that satisfies two properties, which we refer to as *Stability* and *Validity* of the tie-breaking rule respectively. We call the mechanisms underlying this framework *Fencing mechanisms.*

The mechanisms we design can be put in the following general framework: Given a bid vector as input, we search for a pair of sets L, U, where $L \subseteq U \subseteq \mathcal{A}$, that meets the criteria of Stability we define below and then we choose one of the allocations in this restriction according to a valid tie-breaking rule. In Section 5 we show that if the underlying cost-sharing scheme is Fence Monotone, then there exists a unique stable pair at every bid vector.

If the search for the stable pair is exhaustive, then the resulting algorithm runs in exponential time. Given an arbitrary cost sharing scheme that satisfies Fence Monotonicity we do not know of any polynomial time algorithm for computing the stable pair at every input. However, if we restrict our attention to payments that satisfy stronger conditions like, for example, cross-monotonicity we can come up with a polynomial-time algorithm for finding a stable pair.

Definition 2 (Stability) A pair L, U is *stable* at b w.r.t. a cost-sharing scheme ξ if the following conditions are true:

1. For all $i \in L$, $b_i > \xi^*(i, L, U)$,
2. for all $i \in U \setminus L$, $b_i = \xi^*(i, L, U)$, and
3. for all $R \subseteq \mathcal{A} \setminus U$, there is some $i \in R$, such that $b_i < \xi^*(i, L, U \cup R)$.

The first Stability condition ensures that each player in L can be serviced with strictly positive utility. The second Stability condition implies that every player in $U \setminus L$ can be serviced in at least one outcome but with zero utility. The last property requires that if we try to enlarge U, then at least one of the newly added players cannot pay in any possible outcome.

After identifying a stable pair these mechanisms output a set S, where $L \subseteq S \subseteq U$ given by a tie-breaking function.

Definition 3 (Validity) The mapping $\sigma : 2^{\mathcal{A}} \times 2^{\mathcal{A}} \times \mathbb{R}^n \rightarrow 2^{\mathcal{A}}$ is a *valid* tie-breaking rule w.r.t. a cost-sharing scheme ξ, if for all $L \subseteq U \subseteq \mathcal{A}$, the set $S = \sigma(L, U, b)$ satisfies $L \subseteq S \subseteq U$ and for all $i \in S$, $\xi(i, S) = \xi^*(i, L, U)$.

The dependence on the bid vector allows the mechanism of our framework to change its tie-breaking rule between two bid vectors that share a common stable pair. Obviously, condition (a) of Fence Monotonicity guarantees that a valid tie-breaking rule always exists.

Definition 4 We will say that a mechanism (O, p) is a *Fencing Mechanism* if and only if

1. there is a cost-sharing scheme ξ that satisfies Fence Monotonicity such that for all i, $p_i(b) = \xi(i, O(b)))$ and
2. for any bid vector b, $O(b) = \sigma(L, U, b)$, where L, U is a stable pair at b and σ is a valid tie-breaking rule w.r.t. ξ.

It is easy to verify that every Fencing Mechanism satisfies VP from Stability and valid tie-breaking. Moreover, it satisfies CS, because if a player bids higher than any of her payments, then again by Stability she belongs to the set L and gets serviced.

Remark 1 Assume that ξ is cross-monotonic and let S be the output of the Moulin mechanism for some bid vector b. Then, the pair L, U, where $L = \{i \in S \mid b_i > \xi(i, S)\}$ and $U = S$, is the unique stable pair at b. Moreover, the tie breaking rule $\sigma(L, U, b) = U$ is always valid. Therefore, Moulin mechanisms can be viewed as a special case of Fencing mechanisms.

The next theorem completes our characterization by showing that given a cost-sharing scheme that satisfies Fence monotonicity, then Stability and Validity are sufficient and necessary conditions for the design of groupstrategyproof mechanisms. The proof of this theorem is given in Section 5.

Theorem 2 *A mechanism is GSP if and only if it is a Fencing Mechanism.*

4 Every GSP Mechanism is a Fencing Mechanism

4.1 Necessity of Fence Monotonicity

Here, we prove that the cost-sharing scheme of any GSP mechanism satisfies Fence Monotonicity. Let (O, p) be an arbitrary GSP mechanism, let ξ be the corresponding cost-sharing scheme and consider some $U \subseteq \mathcal{A}$. We show that for every $L \subseteq U$, ξ satisfies each one of the Fence Monotonicity conditions using induction on $|U \setminus L|$.

Induction Base: There is only one set $L = U$ under this restriction which implies that $\xi^*(i, L, U) = \xi(i, U)$ for all $i \in U$, hence the set U satisfies properties (a) and (b). For property (c) consider some C and $j \in C$ such that $\xi(j, C) < \xi^*(i, L, U)$. Then, we can set $T = U \setminus C$ since for all $i \in T$, $\xi(i, C \cup T) = \xi(i, U) = \xi^*(i, L, U)$.

Induction Step: First, we define the notion of a *harm relation* and we prove that it is a strict partial order.

Lemma 1 *Consider two pairs $L \subseteq U$ and $L' \subseteq U'$. If $U' \subseteq U$ and $L \subseteq L'$, then for all $i \in U$, $\xi^*(i, L', U') \geq \xi^*(i, L, U)$.*

Proof It is obvious that the minimum payment of player i can only decrease as the set of outcomes, over which the minimum in the definition of ξ^* is taken, becomes larger. □

Definition 5 (Harm relation) Fix two sets $L \subseteq U$ and suppose that $i, j \in U$. We say that *i harms j*, if $\xi^*(j, L, U) < \xi^*(j, L \cup \{i\}, U)$. If i does not harm j, then $\xi^*(j, L, U) = \xi^*(j, L \cup \{i\}, U)$ (we get this equality by applying Lemma 1).

Notice that if $i \in L$ it cannot harm any other element by definition.

Lemma 2 *If i harms j at L, U then for all S_j such that $L \subseteq S_j \subseteq U$ and $\xi(j, S_j) = \xi^*(j, L, U)$ it holds that $\xi(i, S_j \cup \{i\}) = \xi^*(i, L, U)$.*

Proof Assume that *i harms j*, that is $\xi^*(j, L, U) < \xi^*(j, L \cup \{i\}, U)$. Notice that that $i \notin S_j$, since $\xi^*(j, L \cup \{i\}, U) > \xi(j, S_j)$. It follows that $S_j \subset U$. By induction hypothesis we can apply condition (c) of Fence Monotonicity at $L \cup \{i\}, U$. Since the only non-empty subset of $(L \cup \{i\}) \setminus S_j$ is $\{i\}$ we can only set $T = \{i\}$. Therefore, we get that $\xi(i, S_j \cup \{i\}) = \xi^*(i, L \cup \{i\}, U)$ or equivalently $\xi(i, S_j \cup \{i\}) = \xi^*(i, L, U)$

Lemma 3 *(i) The harm relation satisfies anti-symmetry and transitivity and consequently it is a strict partial order and the induced sub-graph $G[U \setminus L]$ is a directed acyclic graph.*

(ii) For every $i \in L$, one of the following holds: either every $j \in U \setminus L$ harms *i or there exists some* sink *k of the subgraph $G[U \setminus L]$ that* does not harm *i.*

(iii) For every $i \in U \setminus L$, one of the following holds: either i is a sink of the sub-graph $G[U \setminus L]$ or there exists some sink k of the sub-graph $G[U \setminus L]$ that is harmed by i.

Proof (i) We show this using the induction hypothesis at every $L \cup \{i\}, U$ for every $i \in U \setminus L$ and more specifically condition (c) of Fence Monotonicity.

If i and j belong to L neither can harm the other as we argued before. Also in the case where some $i \in U \setminus L$ harms some $j \in L$, anti-symmetry follows right away since j does not harm i. Last, consider two distinct elements $i, j \in U \setminus L$ such that i *harms* j. From the definition of ξ^* there is a set S_j, where $j \in S_j$, $L \subseteq S_j \subseteq U$ and $\xi(j, S_j) = \xi^*(j, L, U)$. Using Lemma 2 we get that $\xi(i, S_j \cup \{i\}) = \xi^*(i, L, U)$. Since $L \cup \{j\} \subseteq S_j \cup \{i\} \subseteq U$ we also get that $\xi^*(i, L \cup \{j\}, U) \leq \xi(i, S_j \cup \{i\})$. Putting the last two relations together and applying Lemma 1 we get $\xi^*(i, L \cup \{j\}, U) = \xi^*(i, L, U)$, which means that j does not harm i.

We show transitivity of the harm relation in a similar manner. Consider three distinct players i, j and k, where $i, j \in U \setminus L$ and $k \in U$ (k may belong to L) such that i *harms* j and j *harms* k. Assume towards a contradiction that i *does not harm* k. Since $\xi^*(k, L, U) = \xi^*(k, L \cup \{i\}, U)$, there is some set S_k, where $k \in S_k$, $L \cup \{i\} \subseteq S_k \subseteq U$ such that $\xi(k, S_k) = \xi^*(k, L, U)$. By applying Lemma 2 we get that $\xi(j, S_k \cup \{j\}) = \xi^*(j, L, U)$. As $L \cup \{i\} \subseteq S_k \cup \{j\} \subseteq U$ we reach a contradiction our assumption that $\xi^*(j, L, U) < \xi^*(j, L \cup \{i\}, U)$.

(ii) Suppose that there is some $j \in U \setminus L$ that *does not harm* i. If j is a sink of $G[U \setminus L]$, setting $k = j$ completes our proof. Otherwise, there must be a path that goes through j but does not stop there. Let k be the sink of this path (dag). Notice that transitivity implies that j harms k. Thus, it is impossible that k harms i, since using transitivity again we would deduce that j harms i contradicting our assumption.

(iii) If i is not a *sink* of $G[U \setminus L]$, there must be an path starting from i, which ends at a sink k of this graph. Transitivity implies that i *harms* k. □

Lemma 3 allows the effective use of the induction hypothesis in order to identify a successful coalition if some part is violated. We continue by revealing several very interesting and important allocation properties that are satisfied by any groupstrategyproof mechanism, which we will use to prove the induction step for each property of Fence Monotonicity.

Condition (a) of Fence Monotonicity

For the proof of condition (a) we consider bid vectors where all the players in L have bid b_i^*, all players in $\mathcal{A} \setminus U$ have bid -1, and the players in $U \setminus L$ have bid exactly $\xi^*(i, L, U)$.

We first prove a weaker version of condition (a) of Fence Monotonicity. (Notice that here not all players in S_j but just one player $j \in L$ and the players in $S_j \setminus L$ are guaranteed to pay their minimum payment).

Lemma 4 *For every $j \in L$, there is some set S_j, where $L \subseteq S_j \subseteq U$ such that for all $i \in S_j \setminus L$, $\xi(i, S_j) = \xi^*(i, L, U)$ and $\xi(j, S_j) = \xi^*(j, L, U)$.*

Proof Consider some $j \in L$. We take two cases as Lemma 3(ii) indicates:

Case 1: Suppose that every $i \in U \setminus L$ *harms* j. This implies that $\xi^*(j, L, U) = \xi(j, L)$ and hence we can set $S_j = L$, and Lemma 4 is trivially satisfied, since $S_j \setminus L = \emptyset$.

Case 2: Consider some sink k that *does not harm* j. Applying the induction hypothesis, part (a) of Fence Monotonicity is satisfied at $L \cup \{k\}, U$, and there is a set S, where $L \cup \{k\} \subseteq S \subseteq U$ such that for all $i \in S$, $\xi(i, S) = \xi^*(i, L \cup \{k\}, U)$. Using the fact that k is a sink and also *does not harm* j we get that for all $i \in S \setminus L$, $\xi^*(i, L \cup \{k\}, U) = \xi^*(i, L, U)$ and $\xi^*(j, L \cup \{k\}, U) = \xi^*(j, L, U)$. As a result, we can set $S_j = S$. $\square$

Using this we can prove Lemma 5, which implies that the cost-sharing scheme satisfies condition (a) of Fence Monotonicity.

Lemma 5 *At the bid vector b, where for all $i \in L$, $b_i = b_i^*$, for all $i \in U \setminus L$, $b_i = \xi^*(i, L, U)$, and for all $i \notin U$, $b_i = -1$, it holds that $L \subseteq O(b) \subseteq U$ and for all $i \in O(b)$, $\xi(i, O(b)) = \xi^*(i, L, U)$.*

Proof By CS and VP the output of the mechanism satisfies $L \subseteq O(b) \subseteq U$. First note that a player $i \in U \setminus L$ has utility zero: she is either not serviced, or by VP, if she is serviced her payment cannot exceed her bid and cannot be less than her minimum payment $\xi^*(i, L, U)$ so $\xi(i, O(b)) = b_i = \xi^*(i, L, U)$. It remains to show that every $j \in L$ also pays $\xi^*(j, L, U)$.

Suppose towards a contradiction that for some player $j \in L$, $\xi(j, O(b)) > \xi^*(j, L, U)$ (If the set $U \setminus L$ is empty $U = L$ it is impossible that some $j \in L$ is charged more than $\xi^*(j, L, U) = \xi(j, L)$). Let S_j be the set that is guaranteed to exist by Lemma 4. Then players in $(U \setminus L) \cup \{j\}$ could form a coalition in the following way: the players in $S_j \setminus L$ would bid b_i^* and the players in $U \setminus S_j$ bid -1. By CS and VP the outcome of the mechanism is S_j. Since for every $i \in S_j \setminus L$, $\xi(i, S_j) = \xi^*(i, L, U)$ their utilities remain zero after this manipulation, while the utility of player j strictly increases, and thus the coalition is successful.

Consequently for all $j \in L$, $j \in O(b)$ and $\xi(j, O(b)) = \xi^*(j, L, U)$ and the players in $O(b) \setminus L$ are charged their minimum payment as well. $\square$

As a result, $S = O(b)$ meets the requirements of condition (a) of Fence Monotonicity.

Corollary 1 *Condition (a) of Fence Monotonicity is satisfied at L, U.*

Condition (b) of Fence Monotonicity

Using Lemma 3(iii), we can split the proof in two cases: First we consider the sinks, as the fulfillment of the second condition of Fence Monotonicity for a sink is an immediate consequence of the induction hypothesis.

Lemma 6 *For every sink i of $G[U \setminus L]$ condition (b) of Fence Monotonicity is satisfied at L, U.*

Proof Using the induction hypothesis at $L \cup \{i\}, U$ we get from part (a) of Fence Monotonicity that there is a set S, where $L \cup \{i\} \subseteq S \subseteq U$, such that for all $j \in S$, $\xi(i, S) = \xi^*(j, L \cup \{i\}, U)$. Since i is a sink, she does not harm any other player from $U \setminus L$. So, we have that for all $j \in S \setminus L$, $\xi^*(j, L, U) = \xi^*(j, L \cup \{i\}, U)$. Thus, setting $S_i = S$ condition (b) of Fence Monotonicity is satisfied for i at L, U. □

We already know that condition (b) of Fence Monotonicity is satisfied for sinks at L, U and the induction hypothesis also guarantees that it is satisfied for every player at $L \cup \{i\}, U$, i.e., if we add to L some player $i \notin L$. To use this fact in order to prove Fence Monotonicity for all other cases as well, we prove that if property (b) of Fence Monotonicity is satisfied just for one player i at some L', U', then if the input is b (as defined in the next lemma), this player receives service and is charged $\xi^*(i, L', U')$. (We use L', U' to stress out that these are arbitrary sets not necessarily connected to the sets L, U of the induction step; specifically we are going to apply this lemma for the pairs L, U and $L \cup \{j\}, U$ where j is a sink of the induced graph $G[U \setminus L]$.)

Lemma 7 *Consider some $L' \subseteq U' \subseteq \mathcal{A}$. Suppose that the set S_i as in condition (b) of Fence Monotonicity exists for some player $i \in U' \setminus L'$. Then at any bid vector b with $b_j = b_j^*$ for $j \in L'$, $b_i > \xi^*(i, L', U')$, $b_j = \xi^*(j, L', U')$ for all $j \in U' \setminus (L' \cup \{i\})$, and $b_j = -1$ for all $j \notin U$, it holds that $i \in O(b)$ and $\xi(i, O(b)) = \xi^*(i, L', U')$.*

Proof Like in the proof of Lemma 5 we can show that By CS and VP the output of the mechanism satisfies $L' \subseteq O(b) \subseteq U'$. Moreover, all players in $U' \setminus (L' \cup \{i\})$ have utility zero. We need to show that i is serviced and charged $\xi^*(i, L, U)$.

Suppose towards a contradiction that player i, is either not serviced or is charged $\xi(i, O(b)) > \xi^*(i, L, U)$. Obviously if she is serviced at a higher payment then she prefers the set S_i to the current outcome. If she is not serviced she again prefers S_i, as we assumed that $b_i > \xi^*(i, L, U) = \xi(i, S_i)$ and thus her utility would become positive if the outcome was S_i. In a similar manner like in the proof of Lemma 5 she could form a coalition, with the players in $U' \setminus L'$, enforcing the output S_i. Again our assumption about the payments of the rest of the agents in $S_i \setminus L'$ implies that their utility remains unchanged thus this coalition is successful. □

Now consider a player i in $U \setminus L$, that is *not a sink* of $G[U \setminus L]$ and let j be one of its *sinks* such that i *harms* j (i.e. $i \neq j, k$). In order to prove that property (b) of Fence Monotonicity is satisfied for i, we will invoke group-strategyproofness at certain bid vectors (trying to generalize Example 2 in the Appendix where i takes the role of player 3 and j the role of player 4).

Let $\epsilon > 0$ be an arbitrarily small quantity. Specifically let

$$\epsilon < \min_{\{i,S,S':\xi(i,S)\neq\xi(i,S')\}} |\xi(i,S) - \xi(i,S')|,$$

i.e., a strictly positive quantity that is strictly less than any strictly positive difference between the payments that one player may be charged. Notice that in the case where every player is charged the same amount in every outcome then there is no restriction for ϵ. However, this is not true in the case that we study since i harms j.

We gradually reason about the allocation of the bid vectors $b^j, b^{i,j}$ and finally b^i, which are defined in the following table.

bids	$k \in L$	i	sink j	$k \in U \setminus (L \cup \{i,j\})$	$k \in \mathcal{A} \setminus U$
b^j	b_k^*	$\xi^*(i,L,U)$	$\xi^*(j,L,U)+\epsilon$	$\xi^*(k,L,U)$	-1
$b^{i,j}$	b_k^*	$\xi^*(i,L,U)+\epsilon$	b_j^*	$\xi^*(k,L,U)$	-1
b^i	b_k^*	$\xi^*(i,L,U)+\epsilon$	$\xi^*(j,L,U)+\epsilon$	$\xi^*(k,L,U)$	-1

The allocation properties that we show for the bid vectors we just defined are summarized in the following Table.

	bid	payment of j	payment of the sink k
Claim 1	b^j	$j \notin O(b^j)$	$\xi(j,O(b^j)) = \xi^*(j,L,U)$
Claim 2	$b^{i,j}$	$\xi(i,O(b^{i,j})) = \xi^*(i,L,U)$	$\xi(j,O(b^{i,j})) > \xi^*(j,L,U)$
Claim 3	b^i	$\xi(i,O(b^i)) = \xi^*(j,L,U)$	$j \notin O(b^i)$

Claim 1 At the bid vector b^j the following hold:
(a) player j is serviced and charged $\xi^*(j,L,U)$,
(b) player i is not serviced.

Proof We get that player j is serviced and charged $\xi^*(j,L,U)$ by applying Lemmas 6 and 7 for j with $L' = L$ and $U' = U$.

(b) Suppose towards a contradiction that $i \in O(b^j)$. The payment of player j would be lower bounded by $\xi^*(j, Ł \cup \{i\}, U)$, since $L \cup \{i\} \subseteq O(b^j) \subseteq U$, which contradicts with the fact that j is charged $\xi^*(j,L,U)$, which is strictly lower by our assumption that j *harms* k. □

Claim 2 At the bid vector $b^{j,k}$ the following hold
(a) Player i is serviced and charged $\xi^*(i,L,U)$.
(b) Player j is serviced and charged more than $\xi^*(j,L,U)$.

Proof (a) Since j is a sink of $G[U \setminus L]$, we have that for all $k \in U \setminus (L \cup \{i,j\})$, $b_k^{i,j} = \xi^*(k,L,U)\xi^*(k,L\cup\{j\},U) = \xi^*(k,L,U)$ and $b_i^{i,j} > \xi^*(i,L,U) = \xi^*(j,L\cup\{k\},U)$. Hence, we can apply Lemma 7 for i with $L' = L \cup \{j\}$ and $U' = U$, since condition (b) is satisfied for i at this pair (induction hypothesis) and the bid vector satisfies the requirements of this Lemma.

As a result, $i \in O(b^{j,k})$ and $\xi(i, O(b^{i,j})) = \xi^*(i, L\cup\{j\}, U) = \xi^*(i,L,U)$.

(b) From part (a) we get that the set $O(b^{i,j})$ satisfies that $L \cup \{i,j\} \subseteq O(b^{i,j}) \subseteq U$, and thus the payment of player j is lower bounded by $\xi^*(j, L \cup \{i\}, U)$. Since j *harms* k we get that $\xi^*(j, L\cup\{i\}, U) > \xi^*(j,L,U)$ completing our proof.

Claim 3 At the bid vector b^i the following hold

(a) Player j is not serviced at b^i.

(b) Player i is serviced at b^i.

(c) $L \subseteq O(b^i) \subseteq U$ and every player $k \in O(b^i) \setminus L$, is charged $\xi^*(k,L,U)$.

Proof (a) Assume that player j is serviced at b^i. Notice that by VP and the definition of ϵ, if j is serviced at b^i then her payment cannot exceed $\xi^*(j,L,U)$. Additionally, notice that the only coordinate b^i differs from $b^{i,j}$ is the bid of player j. Thus, strategyproofness is violated, since from Claim 2 the payment of j decreases.

(b) Suppose that i is not serviced at b^i. Then $\{i,j\}$ can form a successful coalition when true values are b^i bidding b^j, since from Claim 1 the utility of j increases from zero to ϵ and the utility of i is kept to zero (she is not serviced at either input).

(c) By VP and CS we get that $L \subseteq O(b) \subseteq U$, thus the payment of every player k that receives the service is lower bounded by $\xi^*(k,L,U)$. By definition of b^i and VP of the mechanism we get the equality for every player in $O(b^i) \setminus (L \cup \{i\})$. Moreover, from the definition of ϵ, we conclude the same for i.

The set $S_i = O(b^i)$ meets the requirements of condition (b) of Fence Monotonicity player i. This together with Lemma 6 implies the following corollary.

Corollary 2 *Condition (b) of Fence Monotonicity is satisfied at L, U.*

Condition (c) of Fence Monotonicity

To show that the cost-sharing scheme satisfies the third property of Fence Monotonicity, at L, U, we need the induction hypothesis only for showing (as we have already done) that condition (a) of Fence Monotonicity is satisfied at this pair and specifically only the allocation properties of Lemma 5. Now we consider inputs for which we do not get anymore directly from CS, that the players in L surely receive service. The idea is to gradually generalize the bid vectors and characterize the allocation of the mechanism. The following table contains allocation properties that any GSP mechanism satisfies.

Lemma 8 *For every bid vector b, where for all $i \in L$, $b_i > \xi^*(i, L, U)$, for all $i \in U \setminus L$, $b_i = \xi^*(i, L, U)$ and for all $i \notin U$, $b_i = -1$, it holds that $L \subseteq O(b) \subseteq U$ and for all $i \in O(b)$, $\xi(i, O(b)) = \xi^*(i, L, U)$.*

Notice that we require that the bid of every player in L exceeds its minimum payment at L, U, ensuring these players strictly prefer to be serviced at some set in the restriction imposed by L, U. This lemma does not hold anymore. Intuitively a GSP mechanism may have to exclude an indifferent player in favor of another that can strictly benefit by his exclusion from the outcome in order to avoid them from forming a successful coalition.

Proof We will prove our statement by induction on the cardinality of the set $Q = \{i \in L \mid b_i \neq b_i^*\}$.

Base: Since $Q = \emptyset$, we simply apply Lemma 5.

Induction step: For the induction step we will show that $L \subseteq O(b) \subseteq U$ and for all $i \in L$, $\xi(i, O(b)) = \xi^*(i, L, U)$. Then it is easy to see that every $i \in O(b) \setminus L$ must be serviced at $\xi^*(i, L, U)$ by VP and the definition of ξ^*.

By the construction of the bid vector we have that $O(b) \subseteq U$. Consider now some $j \in Q$. The induction hypothesis implies that j is serviced and charged $\xi^*(j, L, U)$ at the bid vector (b_j^*, b_{-j}). Suppose that j is not serviced at b resulting in zero utility. Since $b_j - \xi^*(j, L, U) > 0$ she can misreport b_j^* so as to increase her utility to a strictly positive quantity and thus violate strategyproofness of the mechanism. As a result, j must be serviced at b and hence by strategyproofness j must be serviced at the same payment. This is because if the unilateral change of player j can change her payment without changing his allocation, then in the true values corresponded to the bid vector where j was charged the higher amount she could misreport in order to increase her utility.

Now since j is indifferent between the two outcomes group-strategyproofness requires that the same holds also for the rest of the players. Otherwise if the true value of player j was the one corresponding to the bid vector that was strictly worse for some other player i, then j could form a successful coalition with player i by bidding according to the other bid vector that gives greater utility to player i. Therefore, it must be that every $i \in L \setminus Q$ ($i \in O(b)$ by CS since $b_i = b_i^*$) is charged the same payment in either input i.e. $\xi(i, O(b)) = \xi(i, O(b_j^*, b_{-j})) = \xi^*(i, L, U)$ (induction hypothesis). □

	bids			allocation of a GSP mechanism	
	L	$U \setminus L$	$\notin U$		$\forall i \in O(b)$
Lemma 5	b_i^*	$\xi^*(i, L, U)$	-1	$L \subseteq O(b) \subseteq U$	$\xi(i, O(b)) = \xi^*(i, L, U)$
Lemma 8	$> \xi^*(i, L, U)$	$\xi^*(i, L, U)$	-1	$L \subseteq O(b) \subseteq U$	$\xi(i, O(b)) = \xi^*(i, L, U)$
Lemma 9	$> \xi^*(i, L, U)$	$\in \mathbb{R}$	-1	$O(b) \subseteq U$	$\xi(i, O(b)) \geq \xi^*(i, L, U)$

Table 1 Allocation properties of GSP mechanisms. Every family of inputs we consider is a subset of the previous one.

The last property reveals the following important fact about the allocation of GSP mechanisms: If the bids of all players in a set $L \subseteq U$ are more than their respective minimum payments at L, U, (and the players in $\mathcal{A} \setminus U$ do not want to participate), then a GSP mechanism never excludes a subset of L from the outcome in favor of another serviced player, (i.e. so that another player pays less, because the players from L are not present in the outcome). Loosely speaking the players in L rule out any outcome $C \subset U$ such that there is some $j \in C$ such that $\xi(j, C) < \xi^*(j, L, U)$.

Lemma 9 *For every bid vector b, where for all $i \in L$, $b_i > \xi^*(i, L, U)$, for all $i \in U \setminus L$, $b_i \in \mathbb{R}$ and for all $i \notin U$, $b_i = -1$, it holds that: for all $i \in O(b)$, $\xi(i, O(b)) \geq \xi^*(i, L, U)$.*

Proof Let b^0 be any bid vector satisfying the conditions of Lemma 8, which means that for all $i \in L$, $b_i^0 > \xi^*(i, L, U)$, for all $i \in U \setminus L$, $b_i^0 = \xi^*(i, L, U)$ and for all $i \notin U$, $b_i^0 = -1$. Then we relax the constrains we put on the bids of the players in $U \setminus L$ (the rest of the players bid always according to b^0) and prove that no player becomes serviced at a price strictly less than her minimum payment at L, U.

Consider some bid vector b. Let $Q = \{i \in U \setminus L \mid b_i \neq b_i^0\}$. However, we will need some new definitions here. Let

$$R := \{i \in Q \mid i \in O(b) \text{ and } \xi(i, O(b)) > b_i^0\} \tag{1}$$

When the true values are given by b^0 and all the players announce b instead, the set R represents the set of players who participate in the coalition and *lose utility* (it was zero and becomes strictly negative). Finally, let m represent the number of relations of the form $\xi^*(i, L, U) \leq \xi(i, S) < b_i$ for all $i \in R$, i.e.,

$$m = |\{(i, S) \mid L \subseteq S \subseteq U,\ i \in S \text{ and } i \in R \text{ and } \xi^*(i, L, U) \leq \xi(i, S) < b_i\}| \tag{2}$$

Notice that a set S may contribute more than once to this number and that if $R \neq \emptyset$ then $m > 0$, since for all $i \in R$, there is some S such that $\xi(i, S) = \xi^*(i, L, U) = b_i^0 < \xi(i, O(b)) \leq b_i$, where the last inequality follows from VP.

We are now ready to state our induction argument. We show that the allocation of the mechanism satisfies for all $i \in O(b)$, $\xi(i, O(b)) \geq \xi^*(i, L, U)$ for any bid vector b, where for all $i \notin U \setminus L$, $b_i^0 = b_i$ using induction on $|Q| + m$.

Base: For $|Q| + m = 0$ we have that $b = b^0$ and the allocation property follows from Lemma 8.

Induction step: Assume that there is some $j \in O(b)$ that is charged less than $\xi^*(j, L, U)$. Since by VP $O(b) \subseteq U$, either $j \in L$ which implies that $b_j^0 > \xi^*(j, L, U)$ or $j \in U \setminus L$ and thus $b_j^0 = \xi^*(j, L, U)$. In both cases we have that

$$j \in O(b) \text{ and } \xi(j, O(b)) < \xi^*(j, L, U) \leq b_j^0. \tag{3}$$

We will prove that there exists *at least one* successful coalition, contradicting the assumed group-strategyproofness of the mechanism. Note that $j \notin R$.

Case 1: If $R \subset Q$, we construct the bid vector b', where for every $i \in R$, $b'_i = b_i$ and for every $i \notin R$, $b'_i = b^0_i$. We complete the proof of this case by showing that $(Q \setminus R) \cup \{j\}$ form a coalition when their true values are b' bidding b.

First, we prove that in the truthful scenario every player $i \in Q \setminus R$ has zero utility and her utility remains non-negative after the misreporting (profile b). By using the induction hypothesis at b' (differs from b^0 in $|R| < |Q|$ coordinates) we get that for every $i \in Q \setminus R$ such that $i \in O(b')$ (the rest of the players have obviously zero utility), it holds that $\xi^*(i, L, U) \leq \xi(i, O(b')) \leq b'_i$, where the last inequality follows from VP. Since $i \in Q \setminus R$, we get that $b'_i = b^0_i = \xi^*(i, L, U)$ and thus $b'_i = \xi(i, O(b'))$. Now consider the outcome after the misreporting. Either $i \notin O(b)$ and hence she has zero utility after the misreporting or $i \in O(b)$ and her utility becomes $b'_i - \xi(i, O(b)) = b^0_i - \xi(i, O(b)) \geq 0$, where the last inequality follows from Equation 1 as $i \in Q \setminus R$.

Second, we prove that player j strictly increases her utility. There are two cases for j: If she is serviced at b', then the induction hypothesis implies that she is charged $\xi(j, O(b')) \geq \xi^*(j, L, U)$. From Equation 3 we get that j is serviced after the misreporting and charged $\xi(j, O(b)) < \xi^*(j, L, U)$, thus, her payment decreases and consequently her utility increases. Now if j is not serviced at b', then she has zero utility. From Equation 3 we get that $b^0_j > \xi(j, O(b))$ and since $j \notin R$ it follows that $b^0_j = b'_j$. As a result, her utility increases to $b'_j - \xi(j, O(b)) > 0$.

Case 2: Otherwise if $R = Q$, then first we show that it must be the case that for all $i \in R$, $b_i = \xi(i, O(b))$. Assume that the converse holds, i.e., for some $i \in R$, $b_i > \xi(i, O(b))$. Consider the bid vector $(\xi(i, O(b)), b_{-i})$. If i is serviced then by strategyproofness her payment should be the same. As a result, if the true values are given by $(\xi(i, O(b)), b_{-i})$ she has zero utility.

Notice that by lowering the bid of player i, it holds that the quantity $|Q|+m$ is reduced by at least one, since we do not count the relation $b_i > \xi(i, O(b)) > \xi^*(i, L, U)$ anymore. Thus, by the induction hypothesis player j is either not serviced or she is charged an amount greater than or equal to $\xi^*(i, L, U)$.

We use these facts to show When the true values are given by $(\xi(i, O(b)), b_{-i})$, players i and j can form a successful coalition bidding b: The utility of player i is zero in either scenario, while the utility of player j increases since either her payment increases or if she was not serviced in the truthful scenario her utility becomes $b_j - \xi(j, O(b)) > 0$.

As a result, it must be the case that for all $i \in R$, $b_i = \xi(i, O(b))$. Notice that by VP we have that $O(b) \subseteq U$ (for every $i \in \mathcal{A} \setminus U$, it holds that $i \notin R$ and thus $b_i = b^0_i = -1$). Since $\xi^*(j, L, U) > \xi(j, O(b))$ it must be that $L \not\subseteq O(b)$. We show that when the true values are given by b the players in $L \setminus O(b)$ and in Q can form a successful coalition bidding b^0.

Assume that the true values are given by b. The utility of every $i \in Q$ is zero since they are charged an amount equal to their bids. Obviously, the same holds for all $k \in L \setminus O(b)$.

First, we prove that the utility of every $i \in Q$ remains non-negative after the misreporting. Consider some $i \in Q$ such that $i \in O(b^0)$ (obviously the rest players have zero utility). From Lemma 8 it holds that $\xi(i, O(b^0)) = \xi^*(i, L, U)$ and since $\xi^*(i, L, U) = b_i^0$ $(i \in U \setminus L)$ and $b_i^0 < \xi(i, O(b)) = b_i$, we conclude that her utility increases to $b_i - \xi(i, O(b^0)) > 0$.

Second, we show that the utility of every $k \in L \setminus O(b)$ increases, as it becomes strictly positive. Since $k \notin Q$ we get that $b_k = b_k^0$. Moreover, since $k \in L$ we have $b_k^0 > \xi^*(k, L, U)$ and from Lemma 8 we get that $k \in O(b^0)$ and $\xi(k, O(b^0)) = \xi^*(k, L, U)$. As a result, her utility becomes $b_k - \xi(k, O(b^0)) > 0$. □

Using Lemma 9, it is easy to show that any GSP mechanism satisfies condition (c) of Fence Monotonicity.

Claim 4 Consider any $C \subset U$ such that there is some $j \in C$ with $\xi(j, C) < \xi^*(j, L, U)$. At the bid vector b^c, where for all $i \in C$, $b_i^c = b_i^*$, for all $i \in L \setminus C$, $b_i^c = \xi^*(i, L, U) + \epsilon$, and for all $i \notin C \cup L$, $b_i^c = -1$, it holds that

(a) for all $i \in O(b^c) \setminus C$, $\xi(i, O(b^c)) \geq \xi^*(i, L, U)$,

(b) $C \subset O(b^c) \subseteq L \cup C$ and

(c) for all $i \in O(b^c) \setminus C$, it holds that $\xi(i, O(b^c)) = \xi^*(i, L, U)$

Proof (a) Immediate by applying Lemma 9.

(b) VP and CS imply that $C \subseteq O(b^c) \subseteq L \cup C_j$. From (a) and the premises of the Claim $\xi(j, C_j) < \xi^*(j, L, U)$ we additionally get that C is a strict subset of $O(b^c)$.

(c) From (a) we have that for all $i \in O(b^c) \setminus C$, $\xi(i, O(b^c)) \geq \xi^*(i, L, U)$. However all of these players have bid $\xi^*(i, L, U) + \epsilon$, where ϵ is an arbitrarily small positive number, smaller than any payment difference, and by VP would not receive service if they had to pay higher than their bid. So $\xi(i, O(b^c)) = \xi^*(i, L, U)$. □

Setting $T = O(b^c) \setminus C$ (Claim 4 (b) states that $C \subset O(b^c)$), hence, $T \neq \emptyset$) we can conclude that every GSP mechanism satisfies condition (c) of Fence Monotonicity.

Corollary 3 *Condition (c) of Fence Monotonicity is satisfied at L, U.*

5 The classes of GSP and Fencing Mechanisms coincide

We complete our characterization by proving that Fencing Mechanisms are GSP. Throughout this section we fix a cost-sharing scheme ξ that satisfies Fence Monotonicity, and a valid tie-breaking rule σ for ξ. First, we provide some technical Lemmas regarding Stability and Fence Monotonicity.

Lemma 10 *For every bid vector b and set L with $L \subseteq \mathcal{A}$, there exists a unique set U with $U \supseteq L$, such that for all $i \in U \setminus L$ we have $b_i \geq \xi^*(i, L, U)$ and any other set with the same property is a subset of U. Moreover, the pair L, U satisfies condition 3. of Stability.*

Proof Consider the family of all sets U such that for all $i \in U \setminus L$, $b_i \geq \xi(i, L, U)$. First, notice that the set L always satisfies this property, hence, this family is not empty. Assume towards a contradiction that there exist two distinct sets U_1, U_2 in this family none of which is a subset of the other. Then we show that set $U_1 \cup U_2$ also belongs to this family. Indeed for all $i \in U_1 \setminus L$ we have $b_i \geq \xi^*(i, L, U_1) \geq \xi^*(i, L, U_1 \cup U_2)$ as the minimum payment of player i can only decrease as the set of outcomes, over which the minimum in the definition of ξ^* is taken, becomes larger. A similar inequality holds for the players in $i \in U_2 \setminus L$ completing the proof. As a result, this family contains a unique maximal element.

Now we show that if U is the maximal set with this property, then L, U satisfy property 3. of stability. Consider some non-empty $R \subseteq \mathcal{A} \setminus U$. Assume that for all $i \in R$, $b_i \geq \xi^*(i, L, U \cup R)$. From Lemma 1 we have that for all $i \in U \setminus L$, $b_i \geq \xi^*(i, L, U \cup R)$ and thus we reach contradiction by the maximality of U. □

Lemma 11 *Suppose that L, U is a stable pair at the bid vector b and that S is set with the property that for all $i \in S$ we have that $b_i - \xi(i, S) \geq 0$. (choosing output S does not violate VP.) If (a) $S \not\subseteq U$, or*

(b) there is some $i \in S$ with $\xi^*(i, L, U) > \xi(i, S)$,
there exists some non-empty set $T \subseteq L \setminus S$, such that for all $j \in T$ we have $\xi(j, S \cup T) < b_j$.

Proof We will first show that if $S \not\subseteq U$ then there exists some $i \in S$ such that $\xi^*(i, L, U \cup S) > \xi(i, S)$. Since L, U is stable (property 3.) there exists some $i \in S \setminus U$ such that $b_i < \xi^*(i, L, U \cup S)$ and since from the initial assumption $\xi(i, S) \leq b_i$ we get that $\xi(i, S) < \xi^*(i, L, U \cup S)$

The rest of the proof is the same for both cases (note that in what follows, if $S \subseteq U$ then $S \cup U = U$). Note that S is a strict subset of $U \cup S$, since $L \not\subseteq S$. Applying part (c) of Fence Monotonicity we get that there exists a non-empty $T \subseteq L \setminus S$ such that for all $j \in T$ we have $\xi(j, S \cup T) = \xi^*(j, L, U \cup S) \leq \xi^*(j, L, U)$, where the last inequality is by the definition of ξ^*, since the minimum cannot decrease as the set of outcomes over which it is taken becomes larger. From property 1. of stability and since $T \subseteq L$, for all $j \in T$ we have $\xi^*(j, L, U) < b_j$. Consequently for all $j \in T$ we have $\xi(j, S \cup T) < b_j$. □

Lemma 12 *Suppose that $L \subseteq S \subseteq U$ and that there exists a non-empty $T \subseteq \mathcal{A} \setminus S$ such that for all $i \in T$ we have $b_i \geq \xi(i, S \cup T)$ and that for at least one player from T the inequality is strict. Then L, U is not a stable set at the bid vector b.*

Proof Suppose towards a contradiction that L, U is stable at b and that there exists some $T \subseteq \mathcal{A} \setminus S$, such that for all $i \in T$, $b_i \geq \xi(i, S \cup T)$ and that for at least one player the inequality is strict.

Since $L \subseteq S \cup T \subseteq U \cup T$ from the definition of ξ^* we get that for all $i \in T$ we have $\xi^*(i, L, U \cup T) \leq \xi(i, S \cup T)$.

If $T \subseteq U$, then there exists some $i \in T$ such that $b_i > \xi(i, S \cup T) \geq \xi^(i, L, U) \geq \xi^*(i, L, U)$. Since $T \subseteq \mathcal{A} \setminus L$ this contradicts with stability (condition 2.) of b at L, U.

If $T \not\subseteq U$, then $T \setminus U$ is not empty and for all $i \in T \setminus U$ we have $b_i \geq \xi(i, S \cup T) \geq \xi^*(i, L, U \cup T)$, which contradicts with stability (condition 3.) of b at L, U. □

5.1 Uniqueness and Group-strategyproofness

Lemma 13 *For the inputs where there exists a stable pair, it is unique.*

Proof Assume that there are two stable pairs L_1, U_1 and L_2, U_2. First we show that $L_1 = L_2$. Suppose first that $L_1 \neq L_2$. Due to symmetry we only need to reach a contradiction in the case where $L_2 \not\subseteq L_1$, i.e., there exists some $j \in L_2 \setminus L_1$. .

By Fence Monotonicity (part (a)) there exists a set S_2 such that $L_2 \subseteq S_2 \subseteq U_2$ and $\xi(i, S_2) = \xi^*(i, L_2, U_2)$ for all $i \in S_2$. Notice that for all $\in S_2$, $b_j \geq \xi(j, S_2)$, where strict inequality holds only if $j \in L_2$. The idea is to apply Lemma 11 and get that there exists a non-empty $T \subseteq L \setminus S_2$, such that for all $i \in T$ we have $b_i > \xi(i, S_2 \cup T)$. We will then apply Lemma 12 to show that L_2, U_2 is not stable at b which contradicts our initial assumption.

It only remains to show that we can apply Lemma 11. If $S_2 \not\subseteq U_1$ this is immediate. Suppose that $S_2 \subseteq U_1$. If $j \in L_2 \setminus L_1$ then also $j \in U_1 \setminus L_1$ (since $j \notin L_1$ and $j \in S_2 \subseteq U_1$). Thus, from stability (condition 2.) of L_1, U_1, we get that $b_j = \xi^*(j, L_1, U_1)$ and from stability (condition 1.) of L_2, U_2 and since $j \in L_2$, we get that $b_j > \xi(j, S_2)$. Therefore, we get that $\xi(j, S_2) < \xi^*(j, L_1, U_1)$ and consequently S_2 satisfies the requirements of Lemma 11.

We showed that $L_1 = L_2 = L$. Suppose towards a contradiction that $U_1 \neq U_2$. From Lemma 10 there exists a unique maximal set U such that $b_i \geq \xi^*(i, L, U)$ for all $i \in U$. Consequently U_1, U_2 are subsets of U and at least one of them, say U_1 a proper subset of U. Then the players in $U \setminus U_1$ contradict stability (condition 3.). □

Lemma 14 *Fencing mechanisms are GSP at inputs where a stable pair exists.*

With this we mean that given two bid vectors where there exists a stable pair at both, then the allocation of a Fencing mechanisms satisfies group-strategyproofness locally, i.e., if we just restrict the true values to be either of these vectors and have the agents misreport the other.

Proof Let b and b' be two bid vectors, and let L, U and L', U' be their corresponding unique (from Lemma 13) stable pairs and $O(b)$ and $O(b')$ the corresponding outputs. Assume towards a contradiction that some of the players can form a successful coalition when the true values are b reporting b'.

We will first show that any player i served in the new outcome, $i \in O(b')$, has non-negative utility i.e. $b_i \geq \xi(i, O(b'))$. Take some i that is output in

$O(b')$. If $b_i = b'_i$ then it holds trivially since the mechanism satisfies VP at the outcome $O(b')$. If $b_i \neq b'_i$ then i changes his bid to be part of the coalition and consequently his utility after this coalition is non-negative, which gives $b_i - \xi(i, O(b')) \geq 0$.

The next step is to apply Lemma 11 to show that there exists some non-empty $T \subseteq L \setminus O(b')$ such that for all $i \in T$ we have $b_i > \xi(i, O(b') \cup T)$. If $O(b') \not\subseteq U$ then the premises of the Lemma hold trivially.

Suppose that $O(b') \subseteq U$. For the coalition to be successful the utility of at least one player j increases strictly when the players bid b' consequently $j \in O(b')$ and $b_j - \xi(j, O(b')) > 0$. We will show that $\xi^*(j, L, U) > \xi(j, O(b'))$. If j is not served at b and since $O(b') \subseteq U$ from stability we get that $j \in U \setminus L$ and $b_j = \xi^*(j, L, U) > \xi(j, O(b'))$. If j is served at b then her payment equals $\xi^*(j, L, U)$ by the definition of the mechanism and in order that she profits strictly it must be $\xi(j, O(b')) < \xi^*(j, L, U)$, so we can again apply Lemma 11.

Finally we will show that for all $i \in T$ we have $b_i = b'_i$. After the manipulation the players in T are not serviced, while as $T \subseteq L$ from stability we have that in the truthful scenario the players in T are serviced with positive utility. Consequently the players in T wouldn't have an incentive to be part of the coalition and change their bids.

Putting everything together we get that there exists a $T \subseteq \mathcal{A} \setminus O(b')$ such that for all $i \in T$ we have $b'_i > \xi(i, O(b') \cup T)$, which by Lemma 12 contradicts our initial assumption that L', U' is stable at b'. □

5.2 Existence of Stable Pairs

The next Lemma completes our statement that Fencing Mechanisms are GSP.

Lemma 15 *For every bid vector b there exists a unique stable pair. Consequently, Fencing Mechanisms are GSP.*

Proof Let b_i^* be any value that satisfies $b_i^* > \max_{S \subseteq \mathcal{A}} \xi(i, S)$. We will show that there exists a stable pair at any input b by induction on the number m of coordinates that are less than b_i^*, i.e. on the number $m = |\{i \mid b_i < b_i^*\}|$.

Base: For $m = 0$, we only have to show that there exists a stable pair for the bid vector $(b_1^*, \ldots, b_n^*)$ and $\mathcal{A}, \mathcal{A}$ is a stable pair.

Induction Step: Suppose that if a bid vector has $m-1$ coordinates that are less than b_i^*, then it has a stable pair. We will show that if a bid vector b has m coordinates that are less than b_i^* then it also has a stable pair. We will suppose towards a contradiction that there exists no stable pair at b.

We first need some definitions. Let $L^* := \{i \mid b_i \geq b_i^*\}$ and U^* the corresponding (from Lemma 10) maximal set. Notice that for all $i \in L^*$, $b_i \geq b_i^* > \xi^*(i, L^*, U^*)$ by the definition of b_i^*, which implies that the pair L^*, U^* satisfies the first condition of stability. Moreover, from Lemma 12 we get that the third condition of stability is satisfied for L^*, U^* as well. Thus, as L^*, U^* cannot be stable at b (from our assumption), the set $W = \{i \mid i \in U^* \setminus L^* \text{ and } b_i > \xi^*(i, L^*, U^*)\}$ should be non-empty.

For each $i \in W$ we define a corresponding pair L_i, U_i as follows: The pair L_i, U_i is the unique (by Lemma 13) stable pair of (b_i^*, b_{-i}), which exists by the induction hypothesis.

Claim 5 If L_i, U_i is the stable pair of (b_i^*, b_{-i}), then
(a) $L^* \cup \{i\} \subseteq L_i$ and (b) $U_i \subseteq U^*$.
(c) If $j \in L_i \setminus (L^* \cup \{i\})$ then $j \in W$ and $b_j > \xi^*(L^*, U^*)$.

Proof (a) Using the definition of L^* we get that $L^* \cup \{i\}$ contains all the players who have bid higher than any payment of the mechanism in (b_i^*, b_{-i}). Since L_i is stable at (b_i^*, b_{-i}), each one of these players, who have bid strictly higher than any payment of the mechanism, should be serviced (any value higher than any payment satisfies the definition of CS for Fencing Mechanisms), and if they are serviced they obviously have strictly positive utility, and thus $L^* \cup \{i\} \subseteq L_i$.

(b) The idea is to show that for all $j \in (U^* \cup U_i) \setminus L^*$, $b_j \geq \xi^*(j, L^*, U^* \cup U_i)$ and since we defined U^* be the maximal set with this property, we get that $U_i \subseteq U^*$.

From definition of U^* we have that for all $j \in U^* \setminus L^*$, $b_j \geq \xi^*(j, L^*, U^*) \geq \xi^*(j, L^*, U^* \cup U_i)$, where the last inequality follows from Lemma 1.

Now consider some $j \in U_i \setminus U^*$ $(j \neq i)$. Since L_i, U_i is stable at (b_i^*, b_{-i}) we get that $b_j \geq \xi^*(j, L_i, U_i) \geq \xi^*(j, L^*, U_i \cup U^*)$ by applying Lemma 1, since from (a) $L^* \subset L_i$.

(c) From (a),(b) and Lemma 1 we have that $\xi^*(j, L_i, U_i) \geq \xi^*(j, L^*, U^*)$. As we defined L_i, U_i to be the stable pair at (b_i^*, b_{-i}) and $j \in L_i$, for $j \neq i$ we have $b_j > \xi^*(j, L_i, U_i) \geq \xi^*(j, L^*, U^*)$ and as $j \notin L^*$, we have $j \in W$. □

Claim 6 If there exists no stable pair at b, then for all $i \in W$ we have
(a) $b_i \leq \xi^*(i, L_i, U_i)$, and (b) $L^* \cup \{i\} \subset L_i$.

Proof (a) The pair L_i, U_i is stable at (b_i^*, b_{-i}) (by definition of L_i, U_i) but it is not stable at b (from our assumption that there exists no stable pair at b). Since the two bid vectors differ only on the i-th coordinate we deduce that stability (condition 1. as $i \in L_i$) is not satisfied by the i-th coordinate of b, thus $b_i \leq \xi^*(i, L_i, U_i)$.

(b) From Claim 5 (a) we already have that $L_i \supseteq L^* \cup \{i\}$. Suppose towards a contradiction that $L_i = L^* \cup \{i\}$. From Fence Monotonicity (condition (b)) we get that there exists some S_i, where $L^* \subseteq S_i \subseteq U^*$ and $i \in S_i$ such that for all $j \in S_i \setminus L^*$ we have $\xi(j, S_i) = \xi^*(j, L_i, U_i)$.

The idea is to show that $L_i \subseteq S_i \subseteq U_i$, which implies that $\xi^*(i, L_i, U_i) \leq \xi(i, S_i) = \xi^*(i, L^*, U^*)$. Considering also that $\xi^*(i, L^*, U^*) < b_i$, because $i \in W$, we get then that $\xi^*(i, L_i, U_i) < b_i$, contradicting the inequality we showed in (a).

It only remains to show that $L_i \subseteq S_i \subseteq U_i$. Notice first that $L_i \subseteq S_i$ since $L_i = L^* \cup \{i\}$. Moreover, since for all $j \in S_i \setminus L_i$, it holds that $b_j \geq \xi(j, S_i)$ and the bids of these players are the same at (b_i^*, b_{-i}), stability of L_i, U_i (condition 3) implies that $S_i \subseteq U_i$, because otherwise $S_i \setminus U_i \neq \emptyset$ and its elements would violate condition 3. of stability.

Since $W \neq \emptyset$ there exists some $i \in W$ such that L_i has minimum cardinality, i.e. $i = \arg\min_{i \in W} |L_i|$. By Claim 6 (b) there exists some $j \in L_i \setminus (L^* \cup \{i\})$ and by Claim 5 (c) $j \in W$. We will show that $L_j \subset L_i$ contradicting the choice of i.

Claim 7 If there exists no stable pair at b and $j \in L_i \setminus (L^* \cup \{i\})$ then
(a) The pair L_i, U_i is stable at the bid vector $(b^*_i, b^*_j, b_{-\{i,j\}})$.
(b) $\xi^*(j, L_j, U_j) > \xi^*(j, L_i, U_i)$ and $i \notin L_j$.
(c) If there exists some $k \in L_j \setminus L_i$, then the mechanism is not GSP at inputs where a stable pair exists.

Proof (a) The pair L_i, U_i is stable at bid vector $(b^*_i, b^*_j, b_{-\{i,j\}})$, since L_i, U_i is stable at (b_i, b_{-i}) and since raising the bid of player j ($j \in L_i$ from our initial assumption) to b^*_j does not affect its stability.

(b) From stability of L_i, U_i at (b^*_i, b_{-i}) we get that $b_j > \xi^*(j, L_i, U_i)$, while from part (a) of Claim 6 we get that $\xi^*(j, L_j, U_j) \geq b_j$. Consequently $\xi^*(j, L_j, U_j) > \xi^*(j, L_i, U_i)$.

Supposing towards a contradiction that $i \in L_j$ we also get in a similar way as before that the pair L_j, U_j is stable at $(b^*_i, b^*_j, b_{-\{i,j\}})$. By Lemma 13 these two pairs coincide, which is a contradiction because we just showed that the payment of j is different.

(c) We will show that if there exists some $k \in L_j \setminus L_i$ then the mechanism is not GSP at inputs, where a stable pair exists. Observe that $k \neq i, j$.

Consider first the vector $b^{i,j} := (b^*_i, b^*_j, b_{-\{i,j\}})$, which stable pair is L_i, U_i from part (a). As $k \notin L_i$ either k is not serviced, or if k is serviced, then her utility is zero. As for j she is serviced with payment $\xi^*(j, L_i, U_i) < \xi^*(j, L_j, U_j) = \xi(j, O(b^*_j, b_{-j}))$ (from (a)).

Consider then $b^j := (b^*_j, b_{-j})$, where L_j, U_j is stable. As $k \in L_j$ and k is serviced and $b_k > \xi(k, O(b^*_j, b_{-j}))$.

Resuming player j strictly prefers $O(b^{i,j})$ to $O(b^j)$, while for k the situation is exactly the opposite. The idea is to construct a bid vector b' where i has zero utility and either $\{i, j\}$ or $\{i, k\}$ is a successful coalition. Let $b' = (\xi(i, O(b^{i,j})), b^*_j, b_{-\{i,j\}})$ and notice that induction hypothesis implies that there is a stable pair at b'. Moreover, observe that the three bid vectors differ only on the bid of player i and consequently from strategyproofness, at every input i is served, she is charged $\xi(i, O(b^{i,j}))$. This implies that if i is served at b' she is charged an amount equal to her bid. Moreover, i is serviced at b^j in the degenerate case where $b' = b^j$ as otherwise VP would imply that $\xi(j, O(b^j)) \leq b^j_j < b^{i,j}_j = \xi(j, O(b^{i,j}))$. In every case we have that i has zero utility at b' and b^j.

Observe first that k must be serviced at b' and charged $\xi(k, O(b')) \leq \xi(k, O(b^j))$ as otherwise $\{i, k\}$ would have been able to form a successful coalition when the true values are b' bidding b^j. Similarly we can show that $\xi(j, O(b')) \leq \xi(j, O(b^{i,j}))$, excluding the degenerate case $b' = b^j$, because otherwise $\{i, j\}$ would have been able to form a successful coalition when the true values are b' bidding $b^{i,j}$ (the utility of i is kept to zero as in the truthful scenario).

As a result players j and k strictly prefer $O(b')$ to $O(b^j)$ and $O(b^{i,j})$ respectively. If $i \in O(b')$ then $\{i,k\}$ when true utilities are given by $b^{i,j}$ can form a successful coalition bidding b' strictly increasing the utility of k, while keeping the utility of i constant, since she is still served with he same payment. If $i \notin O(b')$ then we deduce that $\{i,j\}$ when the true utilities are given by b^j can form successful coalition bidding b' strictly increasing the utility of j, while keeping the utility of i to zero. □

Now since $j \in W$ and from Claim 7 $i \notin L_j$ and for every $k \in L_j$ we have that also $k \in L_i$ (otherwise Claim7 (c) contradicts Lemma 14), we get that $L_j \subset L_i$, which completes the proof. □

5.3 Necessity of Stability and Valid tie-breaking.

Next, we show that the allocation of every GSP mechanism satisfies Stability and uses a Valid tie-breaking rule. Since its cost-sharing scheme must satisfy Fence Monotonicity, we can prove the following generalization of Lemma 8.

Lemma 16 *Let $L \subseteq U \subseteq \mathcal{A}$. For every bid vector b such that L,U is stable, it holds that $L \subseteq O(b) \subseteq U$ and for all $i \in O(b)$, $\xi(i,O(b)) = \xi^*(i,L,U)$.*

Proof Notice that the only difference between the bid vectors we consider here and the ones we considered in Lemma 8 is that there every $i \in \mathcal{A} \setminus U$ could only bid -1. Our proof here uses again a similar induction on $|Q|$, where $Q = |\{i \in L \mid b_i \neq b_i^*\}|$. We show that $L \subseteq O(b) \subseteq U$ and for all $i \in O(b)$, $\xi(i,O(b)) = \xi^*(i,L,U)$.

Base: First, by CS we have that $L \subseteq O(b)$. Next, we show that no player from $\mathcal{A} \setminus U$ is serviced. Suppose towards a contradiction that $O(b) \setminus U \neq \emptyset$, and let $R := O(b) \setminus U$ the subset of players of $\mathcal{A} \setminus U$ who receive service. We will show that VP is violated. Because L,U are stable, there is some $i \in R$, such that $b_i < \xi^*(i,L,U \cup R) \leq \xi(i,O(b))$ (since $L \subseteq O(b) \subseteq U \cup R$) and thus we reach a contradiction.

As a result, it holds that $L \subseteq O(b) \subseteq U$. Thus, for every $i \in O(b)$, we have that $\xi(i,O(b)) \geq \xi^*(i,L,U))$. By VP we get the equality for every $i \in O(b) \setminus L$. Now assume that the previous inequality is strict for some $j \in L$. We show that the players in $(\mathcal{A} \setminus U) \cup \{j\}$ can form a successful coalition, where every $i \notin U$ announces -1. The utilities of these players remain zero after the misreporting and applying Lemma 8 we get that j is serviced and charged $\xi^*(j,L,U) < \xi(j,O(b))$.

Induction Step: We show that $L \subseteq O(b)$ and for all $i \in L$, $\xi(i,O(b)) = \xi^*(i,L,U)$ exactly like in the proof of Lemma 8. If some $i \in Q$ was not serviced or was charged an amount higher than $\xi^*(i,L,U)$, then she could report b_i^*. According to the induction hypothesis at (b_i^*, b_{-i}) she is serviced and charged $\xi^*(i,L,U)$ which violates strategyproofness.

As a result $Q \subseteq O(b)$ and together with CS we get that $L \subseteq O(b)$. Hence, the payment of every $i \in L$ is at least $\xi^*(i,L,U)$ (we have already shown that this is equal for players in Q). If some player $i \in L \setminus Q$ is charged a higher amount, then she can form a successful coalition with any player in $j \in Q$, where j will report b^*_j and according to the induction hypothesis the payment of player i will decrease to $\xi^*(i, L, U)$.

Now since $L \subseteq O(b)$ in a similar way to the induction base we can show that $U \supseteq O(b)$. Thus, this restriction together with the stability of L, U, implies that for every $i \in O(b)$, $\xi(i, O(b)) = \xi^*(i, L, U)$. □

Given any GSP mechanism (O, ξ) Theorem 1 implies that ξ satisfies Fence monotonicity. Also, Lemmas 15 and 16 imply that $L \subseteq O(b) \subseteq U$ where L, U is the unique stable pair at b and also for all $i \in O(b)$, $\xi(i, O(b)) = \xi^*(i, L, U)$. Hence, we can construct a valid tie-breaking rule σ for ξ such that the mechanism coincides with the Fencing mechanism induced by ξ, σ.

6 Budget Balance and Complexity

We demonstrate the use of our characterization by showing that even in the case of three players GSP mechanisms fail in having a constant budget balance.

Proposition 1 *Let ξ be a cost sharing scheme*

(i) If it satisfies condition (a) of Fence Monotonicity for all L, U with $|U \setminus L| = 1$, then it satisfies semi-cross-monotonicity.

(ii) If it satisfies conditions (b) of Fence Monotonicity for all L, U with $|U \setminus L| = 2$ and (c) of Fence Monotonicity for all L, U with $|U \setminus L| = 1$, then it satisfies the following property: For all $S \subseteq \mathcal{A}$ and all distinct $i, j \in S$, if $\xi(j, S \setminus \{i\}) < \xi(j, S)$ then $\xi(i, S \setminus \{j\}) = \xi(i, S)$.

Proof (i) We apply condition (a) of Fence Monotonicity for $L = S \setminus \{i\}$ and $U = S$. It follows that all players $k \in S \setminus \{i\}$ achieve their minimum payment $\xi^*(k, S \setminus \{i\}, S)$ at the same set of serviced players, either all when the serviced set is S, or all when it is $S \setminus \{i\}$.

There are two cases: If there exists some player $j \in S$ such that $\xi(j, S \setminus \{i\}) < \xi(j, S)$, then this minimum payment is achieved when the set of players that receive service is $S \setminus \{i\}$ and consequently $\xi(k, S \setminus \{i\}) \leq \xi(k, S)$ for all $k \in S \setminus \{i\}$. Otherwise if $\xi(j, S \setminus \{i\}) > \xi(j, S)$, then this minimum payment is achieved when the set of players that receive service is S and consequently $\xi(k, S \setminus \{i\}) \geq \xi(k, S)$ for all $k \in S \setminus \{i\}$.

(ii) We will first use condition (b) of Fence Monotonicity to show that $\xi(i, S \setminus \{j\}) \leq \xi(i, S)$. Let $L = S \setminus \{i, j\}$ and $U = S$. As we assumed that $\xi(j, S \setminus \{i\}) < \xi(j, S)$, it follows that $\xi^*(j, L, U) = \xi(j, S \setminus \{i\})$, (since $S \setminus \{i\}$ and S are the only sets, within the restriction of the mechanism we consider, that contain j). From condition (b) of Fence Monotonicity there must be a set S_i with $L \subseteq S_i \subseteq U$, such that $i \in S_i$ and for all $k \in S_i \setminus L$, $\xi(k, S_i) = \xi^*(k, L, U)$. Since $\xi^*(j, L, U) = \xi(j, S \setminus \{i\})$, it is impossible that $S_i = S$ and consequently $S_i = S \setminus \{j\}$. As player i achieves her minimum payment at $S \setminus \{j\}$ we get that $\xi(i, S \setminus \{j\}) \leq \xi(i, S)$.

Now we will use condition (c) of Fence Monotonicity to show that $\xi(i, S \setminus \{j\}) \geq \xi(i, S)$. Let $L = S \setminus \{j\}, U = S, C \cup \{j\} = S \setminus \{i\}$. Now $\xi(j, S \setminus \{i\}) < \xi(j, S) = \xi^*(j, L, U)$. Since $L \setminus (S \setminus \{i\}) = \{i\}$, $T = \{i\}$ and condition (c) of Fence Monotonicity implies that $\xi(i, S \setminus \{i\} \cup T) = \xi^*(i, L, U)$, i.e. $\xi(i, S) = \xi^*(i, S \setminus \{j\}, S)$. Consequently $\xi(i, S) \leq \xi(i, S \setminus \{j\})$. Putting both inequalities together we get $\xi(i, S) = \xi(i, S \setminus \{j\})$. □

Proposition 1 shows exactly how one can derive the two necessary conditions for group-strategyproofness, which were identified by Immorlica et. al. [6]: Part (i) of this Proposition shows how we can derive semi-cross-monotonicity and part (ii) how we can derive the condition identified in Remark B.1 from [6] (Remark B.1 is a special case of part (ii), namely they showed that if $\xi(i, S \setminus \{j\}) \leq \xi(i, S)$ and $\xi(j, S \setminus \{i\}) \leq \xi(j, S)$, then at most one equality can be strict). The two conditions they identified are obtained if we apply Fence Monotonicity for cases when $U \setminus L$ contains either one or two players, i.e., when we restrict the outcome space to only two or four different possible outcomes.

Theorem 3 *There are cost function families where every GSP mechanism has arbitrarily poor budget balance ratio.*

Proof Let $\mathcal{A} = \{1, 2, 3\}$. Consider the cost sharing function defined on $\mathcal{A}$ as follows: $C(\{1, 2\}) = C(\{1, 3\}) = 1$, $C(\{1\}) = C(\{2\}) = C(\{3\}) = x$, $C(\{2, 3\}) = x^2 + x$ and $C(\{1, 2, 3\}) = x^3 + x^2 + x$, where $x \geq 1$. We show that there is no $\frac{1}{x}$-budget balanced cost sharing scheme, that satisfies Fence Monotonicity.

Assume by contradiction that there is a α-budget balanced cost sharing scheme that satisfies Fence Monotonicity for C where $1 \geq \alpha > \frac{1}{x}$.

First, we consider payments of the players 2 and 3 at the sets $\{1, 2\}$ and $\{1, 3\}$ respectively. From the upper bound of budget balance we have that $\xi(2, \{1, 2\}) \leq 1$ and $\xi(3, \{1, 3\}) \leq 1$. From Proposition 1 (ii) it cannot be the case that both of these players payment increases at the set $\{1, 2, 3\}$. That is either $\xi(2, \{1, 2, 3\}) \leq 1$ or $\xi(3, \{1, 2, 3\}) \leq 1$ (or both). W.l.o.g. we assume that $\xi(2, \{1, 2, 3\}) \leq 1$. We consider now two cases for $\xi(2, \{2, 3\})$.

Case 1: Suppose that $\xi(2, \{2, 3\}) \leq 1$. From the lower bound of budget-balance at the set $\{2\}$, we have that $\xi(2, \{2\}) > 1$, and thus $\xi(2, \{2, 3\}) < \xi(2, \{2\})$. We apply Proposition 1 (ii) and deduce that $\xi(3, \{2, 3\}) \leq \xi(3, \{3\}) \leq C(3) = x$. Thus, the total sum of the shares at the set $\{2, 3\}$ is $\xi(2, \{2, 3\}) + \xi(3, \{2, 3\}) \leq x + 1$, which contradicts our assumption about the budget balance of the mechanism, since $C(\{2, 3\}) = x^2 + x$.

Case 2: Suppose that $\xi(2, \{2, 3\}) > 1 \geq \xi(2, \{1, 2, 3\})$. Notice that the contra-positive implication of Proposition 1 (i) implies that $\xi(3, \{1, 2, 3\}) \leq \xi(3, \{2, 3\}) \leq C(\{2, 3\}) - \xi(2, \{2, 3\}) < x^2 + x - 1$. Moreover, as the payment of player 2 decreases from $\{2, 3\}$ to $\{1, 2, 3\}$, Proposition 1 (ii) implies that the payment of player 1 doesn't increase from $\{1, 3\}$ to $\{1, 2, 3\}$, i.e. $\xi(1, \{1, 2, 3, \}) \leq \xi(1, \{1, 3\}) \leq C(\{1, 3\}) = 1$. Now summing the shares at the set $\{1, 2, 3\}$, we get that $\xi(1, \{1, 2, 3\}) + \xi(2, \{1, 2, 3\}) + \xi(3, \{1, 2, 3\}) \leq$

$x^2 + x + 1$, which is again contradicts our budget balance assumption, since $C(\{1, 2, 3\}) = x^3 + x^2 + x$. □

Notice that Fencing Mechanism run in exponential time, as the only way we know to compute its outcome is to search exhaustively all possible pairs and check if they are stable. In fact, we only need to search the lower set L, since there is a polynomial time algorithm (which is defined in the proof of the next theorem) for checking if there is a set U such that L, U is stable.

A natural question that arises is if it is computationally more efficient to find the appropriate outcome than identifying the stable pair.

Theorem 4 *Suppose that we are given the outcome of a GSP mechanism at b. Given polynomial access to $\xi^*(i, L, U)$ for all $L \subseteq U \subseteq \mathcal{A}$ and all $i \in U$, we can identify the stable pair in polynomial time.*

Proof Consider the following process, which takes as input a bid vector b and a set L.

repeat
  $U \leftarrow L \cup \{i \in U \setminus L \mid b_i \geq \xi^*(i, L, U)\}$
until For all $i \in U \setminus L$, $b_i \geq \xi^*(i, L, U)$

We will prove that if we feed this process with the lower set L of the stable pair at b, then the outcome is the upper set of the stable pair.

Obviously by the definition of the process, for its final set U it holds that for all $i \in U$, $b_i \geq \xi^*(i, L, U)$. First, we show that U is the maximal set with this property.

Assume that there is some U' with $U' \not\subseteq U$, that satisfies this property, then we have that for all $i \in (U \cup U') \setminus L$, $b_i \geq \xi^*(i, L, U \cup U')$. Thus, it is impossible that the players in $U \cup U'$ are removed at any step of the previous process, contradicting our assumption that $U \neq U' \cup U$ is the outcome of this process.

Now consider the upper set U'' of the stable pair at b. Since U'' satisfies this property it follows that $U'' \subseteq U$. Notice that if $U'' \subset U \neq \emptyset$, then the elements of $U \setminus U''$ would violate stability of L, U''. Thus, $U'' = U$ and consequently this process outputs the upper set of the stable pair.

As a result, given an outcome S of a GSP mechanism we can compute $L := \{i \in S \mid b_i > \xi(i, S)\}$ and use this process to find the upper set U. The time-complexity of this algorithm is polynomial in the number of players assuming that we have polynomial-time access to every $\xi^*(i, L, U)$ for all $L \subseteq U \subseteq \mathcal{A}$ and all $i \in U$. □

7 Conclusion and future directions

We believe that the most interesting future directions are the following: How can our characterization be applied for obtaining cost-sharing mechanisms with better budget-balance guarantees or lower bounds for specific problems?

Does there exist a polynomial-time algorithm for finding the allocation of a cost-sharing scheme that satisfies Fence Monotonicity or maybe can we show that the problem of finding a stable pair is computationally hard? Given such a result, how should our characterization be restricted when we add tractability as an additional requirement?

Finally, our characterization could be used as a guidance for characterizing mechanisms for specific cost-sharing problems which can more effectively capture the hardness of the problem.

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A Examples of Mechanisms that violate just one part of Fence Monotonicity and are not GSP

We will give three representative examples to illustrate, why a cost sharing scheme, which does not satisfy Fence Monotonicity, cannot give rise to a group-strategyproof mechanism. We chose our examples in a way that only one condition of Fence Monotonicity is violated and only at a specific pair L, U. (In fact in condition (c), the violation is present at two pairs, however it can be shown that this is unavoidable.)

Example 1 (a)
Let $\mathcal{A} = \{1, 2, 3, 4\}$. We construct a cost sharing scheme, such that condition (a) of Fence Monotonicity is not satisfied at $L = \{1, 2\}$ and $U = \{1, 2, 3, 4\}$, as follows.

ξ	1	2	3	4
$\mathbf{\{1,2,3,4\}}$	**30**	**30**	**30**	**30**
$\mathbf{\{1,2,3\}}$	**20**	**30**	**30**	–
$\mathbf{\{1,2,4\}}$	**30**	**20**	–	**30**
$\{1,3,4\}$	30	–	20	30
$\{2,3,4\}$	–	30	30	20

ξ	1	2	3	4
$\mathbf{\{1,2\}}$	**30**	**30**	–	–
$\{1,3\}$	20	–	30	–
$\{1,4\}$	30	30	–	–
$\{2,3\}$	–	30	30	–
$\{2,4\}$	–	20	–	30

ξ	1	2	3	4
$\{3,4\}$	–	–	30	30
$\{1\}$	30	–	–	–
$\{2\}$	–	30	–	–
$\{3\}$	–	–	30	–
$\{4\}$	–	–	–	30

Consider the bid vector $b := (b_1^*, b_2^*, 30, 30)$. Notice that players 3 and 4 are indifferent to being serviced or not, as the single value they may be charged as payment equals their bid. Moreover, notice that either player 1 or player 2 (or both) must pay 30 strictly over their minimum payment 20 under this restriction. Without loss of generality assume that $\xi(1, O(b)) = 30$. Consider the bid vector $b' := (b_1^*, b_2^*, b_3^*, -1)$. By VP and CS it holds that $O(b') = \{1, 2, 3\}$ and thus $\xi(1, O(b')) < \xi(1, O(b))$. Notice that the utilities of players 3 and 4 remain zero, and consequently $\{1, 3, 4\}$ form a successful coalition. In a similar manner we prove the existence of successful coalition when $\xi(2, O(b)) = 30$.

Example 2 (b)
Let $\mathcal{A} = \{1, 2, 3, 4\}$. We construct a cost sharing scheme, such that condition (b) of Fence Monotonicity is not satisfied at $L = \{1, 2\}$ and $U = \{1, 2, 3, 4\}$ for player 3, as follows.

ξ	1	2	3	4
$\mathbf{\{1,2,3,4\}}$	**30**	**30**	**30**	**30**
$\mathbf{\{1,2,3\}}$	**30**	**30**	**40**	–
$\mathbf{\{1,2,4\}}$	**30**	**30**	–	**20**
$\{1,3,4\}$	30	–	30	30
$\{2,3,4\}$	–	30	30	30

ξ	1	2	3	4
$\mathbf{\{1,2\}}$	**30**	**30**	–	–
$\{1,3\}$	30	–	30	–
$\{1,4\}$	30	–	–	30
$\{2,3\}$	–	30	30	–
$\{2,4\}$	–	30	–	30

ξ	1	2	3	4
$\{3,4\}$	–	–	30	30
$\{1\}$	30	–	–	–
$\{2\}$	–	30	–	–
$\{3\}$	–	–	30	–
$\{4\}$	–	–	–	30

Consider the bid vector $b^{3,4} := (b_1^*, b_2^*, 35, b_4^*)$. Strategyproofness implies that $3 \in O(b^{3,4})$, since otherwise if she is not serviced (zero utility), she can misreport b_3^* changing the outcome to $\{1,2,3,4\}$ and increasing her utility to $35 - 30 > 0$.

Next, consider the bid vector $b^4 := (b_1^*, b_2^*, 30, 25)$. Assume that $4 \notin O(b^4)$. Moreover, notice that by VP it is impossible that $3 \in O(b^4)$. Thus, $\{3,4\}$ can form a successful coalition bidding $b' = (b_1^*, b_2^*, -1, b_4^*)$, changing the outcome to $\{1,2,4\}$ increasing the utility of player 4 to $25 - 20 > 0$, while keeping the utility of player 3 at zero.

Finally, consider the bid vector $b^3 := (b_1^*, b_2^*, 35, 25)$. Notice that the b^3 differs with b^4 and $b^{3,4}$ in the coordinates that correspond to players 3 and 4 respectively. Like in the case of b^4 the only possible outcomes by VP and CS at b^3 are $\{1,2,4\}$ and $\{1,2\}$.

Assume that player 4 is serviced at b^3, which implies that $\xi(4, O(b^4)) < \xi(4, O(b^{3,4}))$. This contradicts strategyproofness, since when the true values are $b^{3,4}$, player 4 can bid according to b^3 in order to decrease her payment and still being serviced.

Now, assume that player 4 is not serviced at b^3. Then $\{3,4\}$ can form a successful coalition when true values are b^3 bidding b^4, increasing the utility of player 4 to $25-20 > 0$, while keeping the utility of player 3 at zero.

Example 3 (c)

Let $\mathcal{A} = \{1,2,3,4\}$. This time we construct a cost sharing scheme, such that part (c) is not satisfied for $L = \{1,2\}$ (or $\{1,2,3\}$) and $U = \{1,2,3,4\}$ and specifically for $C_i = \{3,4\}$ and $i = 3$.

ξ	1	2	3	4
$\mathbf{\{1,2,3,4\}}$	**30**	**30**	**30**	**30**
$\mathbf{\{1,2,3\}}$	**20**	**20**	**30**	–
$\mathbf{\{1,2,4\}}$	**30**	**30**	–	**30**
$\{1,3,4\}$	30	–	30	30
$\{2,3,4\}$	–	30	30	30

ξ	1	2	3	4
$\mathbf{\{1,2\}}$	**20**	**20**	–	–
$\{1,3\}$	20	–	20	–
$\{1,4\}$	30	30	–	–
$\{2,3\}$	–	20	20	–
$\{2,4\}$	–	30	–	30

ξ	1	2	3	4
$\{3,4\}$	–	–	*20*	30
$\{1\}$	30	–	–	–
$\{2\}$	–	30	–	–
$\{3\}$	–	–	30	–
$\{4\}$	–	–	–	30

Suppose that the values are $b := (25, 25, b_3^*, 30)$. The feasible by VP outcomes are $\{1,2,3\}$, $\{1,3\}$, $\{2,3\}$, $\{3,4\}$ and $\{3\}$. Notice that player 4 has zero utility regardless of the outcome.

Assume that $O(b) \neq \{1,2,3\}$ and w.l.o.g that $1 \notin O(b)$. Then $\{1,2,4\}$ can form a successful coalition bidding $b' := (b_1^*, b_2^*, b_3^*, -1)$ increasing the utility of player 1 to $25 > 0$ without decreasing the utility of player 2 (either $O(b) = \{2,3\}$ and $\xi(2, O(b')) = \xi(2, O(b))$ and her utility remains the same utility or $2 \notin O(b)$ and her utility increases to $25 > 20$, like in the case of player 1) and keeping the utility of player 4 at zero.

Finally, assume that $O(b) = \{1,2,3\}$. Then $\{3,4\}$ can form a successful coalition bidding $b'' := (25, 25, b_3^*, b_4^*)$. Obviously $\{3,4\} \subseteq O(b'')$. Notice that VP excludes each of the following outcomes: $\{1,3,4\}$, $\{2,3,4\}$ and $\{1,2,3,4\}$. As a result, $O(b'') = \{3,4\}$ implying that the utility of player 3 increases, as $\xi(3, O(b)) > \xi(3, O(b''))$, while player 4 keeps her utility at zero.

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